Formulae and Tables for Statistical work

FORMULAE AND TABLES FOR STATISTICAL WORK

EDITED BY
C.R. RAO
S.K. MITRA
A. MATTHAI
K.G. RAMAMURTHY

STATISTICAL PUBLISHING SOCIETY

VALUE OF π UPTO 2035 DECIMAL PLACES*

3·14159	26535	89793	23846	26433	83279	50288	41971	69399	37510
58209	74944	59230	78164	06286	20899	86280	34825	34211	70679
82148	08651	32823	06647	09384	46095	50582	23172	53594	08128
48111	74502	84102	70193	85211	05559	64462	29489	54930	38196
44288	10975	66593	34461	28475	64823	37867	83165	27120	19091
45648	56692	34603	48610	45432	66482	13393	60726	02491	41273
72458	70066	06315	58817	48815	20920	96282	92540	91715	36436
78925	90360	01133	05305	48820	46652	13841	46951	94151	16094
33057	27036	57595	91953	09218	61173	81932	61179	31051	18548
07446	23799	62749	56735	18857	52724	89122	79381	83011	94912
98336	73362	44065	66430	86021	39494	63952	24737	19070	21798
60943	70277	05392	17176	29317	67523	84674	81846	76694	05132
00056	81271	45263	56082	77857	71342	75778	96091	73637	17872
14684	40901	22495	34301	46549	58537	10507	92279	68925	89235
42019	95611	21290	21960	86403	44181	59813	62977	47713	09960
51870	72113	49999	99837	29780	49951	05973	17328	16096	31859
50244	59455	34690	83026	42522	30825	33446	85035	26193	11881
71010	00313	78387	52886	58753	32083	81420	61717	76691	47303
59825	34904	28755	46873	11595	62863	88235	37875	93751	95778
18577	80532	17122	68066	13001	92787	66111	95909	21642	01989
38095	25720	10654	85863	27886	59361	53381	82796	82303	01952
03530	18529	68995	77362	25994	13891	24972	17752	83479	13151
55748	57242	45415	06959	50829	53311	68617	27855	88907	50983
81754	63746	49393	19255	06040	09277	01671	13900	98488	24012
85836	16035	63707	66010	47101	81942	95559	61989	46767	83744
94482	55379	77472	68471	04047	53464	62080	46684	25906	94912
93313	67702	89891	52104	75216	20569	66024	05803	81501	93511
25338	24300	35587	64024	74964	73263	91419	92726	04269	92279
67823	54781	63600	93417	21641	21992	45863	15030	28618	29745
55706	74983	85054	94588	58692	69956	90927	21079	75093	02955
32116	53449	87202	75596	02364	80665	49911	98818	34797	75356
63698	07426	54252	78625	51818	41757	46728	90977	77279	38000
81647	06001	61452	49192	17321	72147	72350	14144	19735	68548
16136	11573	52552	13347	57418	49468	43852	33239	07394	14333
45477	62416	86251	89835	69485	56209	92192	22184	27255	02542
56887	67179	04946	01653	46680	49886	27232	79178	60857	84383
82796	79766	81454	10095	38837	86360	95068	00642	25125	20511
73929	84896	08412	84886	26945	60424	19652	85022	21066	11863
06744	27862	20391	94945	04712	37137	86960	95636	43719	17287
46776	46575	73962	41389	08658	32645	99581	33904	78027	59009
94657	64078	95126	94683	98352	59570	98258			

* The computation was carried out by G. W. Reitwiesner on ENIAC using a total of 70 hours of machine running time in July 1949 using the formula $\pi/4 = 4$ arc tan $1/5$—arc tan $1/239$ in conjunction with the Gregory series

$$\text{arc tan } x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} x^{2n+1}.$$

It would be of interest to apply tests of randomness on the decimal digits upto 500, 1000, 1500, 2000 places. For instance, the frequencies of 0, 1, ..., 9 in the first 2000 decimals are

182, 212, 207, 189, 195, 205, 200, 197, 202, 211

which are all close to expected 200. (Apply chi-square tests).

Formulae and Tables for Statistical Work

Formulae and Tables for Statistical Work

EDITED BY

C. RADHAKRISHNA RAO, Sc.D., F.R.S

Director and Secretary

SUJIT KUMAR MITRA, Ph.D

Professor of Statistics

ABRAHAM MATTHAI, D.Phil

Professor of Statistics

AND

K. G. RAMAMURTHY

SQC Specialist

INDIAN STATISTICAL INSTITUTE
CALCUTTA

STATISTICAL PUBLISHING SOCIETY

First Printing 1966

Second Edition 1975

Typeset in Eka Press, 204/1, B. T. Road, Calcutta 700 035 and printed by offset at Dipti Printing and Binding Works, Calcutta 700 004.

PREFACE

The present volume had its origin mainly in the recognition of a need repeatedly brought up by our students and colleagues, as well as by a number of professional statisticians and research workers engaged in various applied fields, for a handbook which is not merely a collection of mathematical and statistical tables but which contains reference material that will aid memory and offer guidance on various points of statistical theory and practice. With this end in view we started on a project some years ago, and have since evolved and incorporated in this volume a method of presenting formulae and tables so as to offer a great facility in their use for statistical data analysis.

Part I of this volume entitled "General Notes and Formulae" provides a fairly comprehensive but selected set of formulae together with brief explanations, under the following heads : (I) moments and cumulants, (II) discrete and (III) continuous distributions, (IV) standard errors, (V) sample survey estimates and standard errors, and (VI) numerical analysis. The formulae, together with related notes, will be found to be a collection in one place of what are usually scattered in different text books, or other sources, and of what are usually required for statistical applications. In the presentation of the material in this section, special attempts have been made to highlight certain aspects (such as the use of interpolation formulae) which the authors have considered important and, at the same time, which have not been adequately discussed elsewhere. The list of discrete distributions given in Part I would be of interest to even research workers in theoretical statistics. In the presentation of the formulae and notes, emphasis has been placed on furnishing necessary guidelines for practical use rather than derivations of proofs.

The sixtyseven tables given in Part II of the volume fall under two broad categories : firstly tables associated with probability distributions and relating directly to tests of significance and other analytic statistical methods, and secondly, tables which find direct use in the processing of statistical data.

A special feature in the presentation of these tables is that, before each table, an explanatory note, giving a description of the table and containing illustrative examples, is provided. Where necessary, the type of formulae to be used for interpolation in the tables and the accuracy attainable are also indicated. Where the nature of interpolation is not indicated, in general, it could be assumed that linear interpolation would suffice. In the explanatory note on each table, a section is devoted to give references to other available publications containing more extensive tables.

Some of the special features of the section on tables are : i) a table of interpolation co-efficients, ii) an expanded table of numerical integration co-efficients, iii) percentage points of the beta distribution so as to give directly the significant

values of the multiple correlation co-efficient, iv) expanded tables for angular transformation of the binomial proportion and z-transformation of the correlation co-efficient, v) a comparatively extensive table of the normal distribution, vi) mathematical tables of a wide variety, vii) tables to facilitate conversion of number systems for special use in programming for electronic computers, viii) a handy arrangement of control chart factors, ix) a collection of tables for lot quality estimation, x) a simplified set of lot acceptance sampling inspection tables, xi) random permutations of digits and random numbers, etc.

It is hoped that the collection of tables and formulae, together with associated notes in this volume, will form a fairly adequate and handy aid to professional statisticians, research workers and others who have to deal with problems involving statistical analysis and inference.

Notation

In Part I (General Notes and Formulae), Roman numerals (I, II,......) are used to number the chapters and lower case Latin alphabet (**a**, **b**,......) for sections. Thus, a reference such as **IIb** means section **b** in chapter **II** of Part **I**.

In Part **II** (Tables with Explanatory Notes), the chapters are numbered as **1**, **2**, ... ; sections as **1**, **2**, ..., and subsections as **a**, **b**, ... Thus a reference such as **15.2b** means the subsection **b** in section **2** of chapter **15**. When a Chapter does not contain sections, references to subsections are made such as **19c**, i.e., subsection **c** in Chapter **19**. Tables in a chapter are numbered serially ; thus, Table 13.2 stands for the second table in chapter **13** of Part **II**.

Calcutta, India
June, 1966

C. R. Rao
S. K. Mitra
A. Matthai

Preface to Second Edition

A number of new tables and explanatory notes useful in Statistical Quality Control (SQC) have been added (W test for normality, tests for outliers, probability plotting, CUSUM charts, tolerance intervals, distribution of ranges etc.). Tables of sampling plans are withdrawn. It is hoped to bring out a separate publication for these tables. The values of π and e are given upto 2035 and 2500 decimal places respectively.

C. R. Rao
S. K. Mitra
A. Matthai
K. G. Ramamurthy

Calcutta, India
June, 1974

Acknowledgements

We are indebted to many of our colleagues in the Indian Statistical Institute for help rendered to us at different stages in the preparation of the "Formulae and Tables". We are specially thankful to Dr. B. Ramachandran and Shri T. S. Krishnan for their help in the final stages of publication, and to Dr. Jogabrata Roy for help in using the various computational and other facilities at the Institute which have made our task relatively less burdensome and accounted for its early completion.

In compiling the numerous statistical and mathematical tables, we have made use of journals, books and other published and unpublished sources, which we gratefully acknowledge here.

We are indebted to the late Sir Ronald A. Fisher, F.R.S., Cambridge and Dr. Frank Yates, F.R.S., Rothamsted, and also to Messrs. Oliver and Boyd Ltd., Edinburgh, for permission to reprint Table I (The normal distribution), Table XXIII (Orthogonal polynomials), Table XV (Latin squares) and Table XVI (Complete sets of orthogonal latin squares), and Table XX (Scores for ordinal or ranked data) from *Statistical Tables for Biological, Agricultural and Medical Research.*

We are indebted to Professor E. S. Pearson and Dr. H. O. Hartley for permission to reprint Table 18 (Percentage points of the F distribution), Table 24 (Percentage points of the extreme standardised deviate from the population mean), Table 26 (Percentage points of the extreme Studentised deviate from the sample mean) and Table 31 (Percentage points of the ratio s^2_{max}/s^2_{min}), from *Biometrika Tables for Statisticians*, Vol. 1.

We are indebted to the Indian Standards Institution for permission to reprint the acceptance sampling plans from their bulletin IS: 2500 (Part I)—1963: *Sampling Inspection Tables.*

We owe a special debt of gratitude to the Statistical Publishing Society, Calcutta, for the keen interest they have shown in the publication of the "Formulae and Tables" and to the Eka Press, Calcutta, for the promptness and accuracy with which they have printed this volume.

EDITORS

CONTENTS

PART II

TABLES WITH EXPLANATORY NOTES

PART I

I. MOMENTS AND CUMULANTS

a. Relation between raw moments (μ'_r) and central moments (μ_r)

For a distribution function $F(x)$ let $\mu'_r = \int_{-\infty}^{\infty} (x-c)^r \, dF$ and $\mu_r = \int_{-\infty}^{\infty} (x-m)^r \, dF$ where $m = \int_{-\infty}^{\infty} x dF$ is the mean value and c is an arbitrary origin. Then

$$\mu_r = \mu'_r - r\mu'_{r-1}\mu'_1 + \binom{r}{2}\mu'_{r-2}\mu_1'^2 - \ldots + (-1)^{r-1}(r-1)\mu_1'^r.$$

Thus,

$$\mu_2 = \mu'_2 - \mu_1'^2$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4.$$

b. Relation between factorial moments and raw moments

The r-th factorial moment about an arbitrary origin c is defined by

$$\mu'_{[r]} = \int_{-\infty}^{\infty} (x-c)(x-c-h)\cdots(x-c-rh+h)dF.$$

If μ'_r be raw moments also about the same origin c, we have the following relations between the factorial and the raw moments.

factorial moments	in terms of raw moments	raw moments	in terms of factorial moments
$\mu'_{[1]}$	μ'_1	μ'_1	$\mu'_{[1]}$
$\mu'_{[2]}$	$\mu'_2 - h\mu'_1$	μ'_2	$\mu'_{[2]} + h\mu'_{[1]}$
$\mu'_{[3]}$	$\mu'_3 - 3h\mu'_2 + 2h^2\mu'_1$	μ'_3	$\mu'_{[3]} + 3h\mu'_{[2]} + h^2\mu'_{[1]}$
$\mu'_{[4]}$	$\mu'_4 - 6h\mu'_3 + 11h^2\mu'_2 - 6h^3\mu'_1$	μ'_4	$\mu'_{[4]} + 6h\mu'_{[3]} + 7h^2\mu'_{[2]} + h^3\mu'_{[1]}$

c. Relation between cumulants and moments

Cumulants are formally defined by the identity

$$\exp\left\{\kappa_1 t + \frac{\kappa_2 t^2}{2!} + \frac{\kappa_3 t^3}{3!} + \ldots\right\} = 1 + \mu'_1 t + \mu'_2 \frac{t^2}{2!} + \mu'_3 \frac{t^3}{3!} + \ldots$$

From the definition it follows that the cumulants, except for the first, are invariant for change of origin.

cumulants	in terms of moments	moments	in terms of cumulants
κ_1	μ_1	μ_1'	κ_1
κ_2	μ_2	μ_2	κ_2
κ_3	μ_3	μ_3	κ_3
κ_4	$\mu_4-3\mu_2^2$	μ_4	$\kappa_4+3\kappa_2^2$
κ_5	$\mu_5-10\mu_3\mu_2$	μ_5	$\kappa_5+10\kappa_3\kappa_2$
κ_6	$\mu_6-15\mu_4\mu_2-10\mu_3^2+30\mu_2^3$	μ_6	$\kappa_6+15\kappa_4\kappa_2+10\kappa_3^2+15\kappa_2^3$

d. Probability and moment generating functions

For a discrete distribution assigning probabilities $p_0, p_1, p_2, \ldots$ to variable values 0, 1, 2, ... consider the following generating functions :

(i) the probability generating function (pgf)

$$P(t) = \sum_{i=0}^{\infty} p_i t^i,$$

(ii) the factorial moment generating function (fmgf)

$$M_f(t) = \sum_{i=0}^{\infty} \mu'_{[i]} t^i / i!,$$

where for the factorial moments $\mu_{[i]}$ the origin $c = 0$ and $h = 1$, and

(iii) the moment generating function (mgf)

$$M(t) = \sum_{i=0}^{\infty} \mu'_i t^i / i!$$

where for the raw moments μ_i the origin $c = 0$

We have here the relations

$$M_f(t) = P(1+t), \quad M(t) = P(e^t)$$

e. Sheppard's correction for grouping

For a distribution function $F(x)$ the proportion of observations in an interval $(a_i, a_{i+1}]$ is given by

$$\pi_i = \int_{a_i}^{a_{i+1}} dF$$

Let the system of intervals $(a_i, a_{i+1}]$ for $i = 0, \pm 1, \pm 2, \ldots$ cover the entire range of the distribution. Consider the grouped frequency distribution with variate values $b_i = \dfrac{a_i + a_{i+1}}{2}$ and relative frequencies π_i. The r-th raw moment of the grouped frequency distribution is represented by $\bar{\mu}'_r = \sum_{i=-\infty}^{\infty} (b_i - c)^r \pi_i$. Cumulants and

factorial moments calculated from the grouped frequency distribution will be similarly indicated by $\bar{\kappa}_r$ and $\bar{\mu}'_{[r]}$ respectively.

Consider the case where intervals are of equal width h and the distribution admits a density function $f(x)$. Assume further that: (a) $f(x)$ and its first $2s$ derivatives are continuous for all x, (b) $x^{k+2}\dfrac{d^i f(x)}{dx^i}$ is bounded for all x and for $i=0, 1, 2, \ldots, 2s$, where k and s are certain positive integers. Under these conditions for all $r \leqslant k$

$$\text{(i)}\quad \mu'_r = \sum_{j=0}^{r} \binom{r}{j} (2^{1-j}-1) B_j h^j \bar{\mu}'_{r-j} + R$$

$$\text{(ii)}\quad \mu'_{[r]} = \sum_{j=0}^{r} \binom{r}{j} B_j^{(j+2)}\left(\frac{3}{2}\right) h^j \bar{\mu}'_{[r-j]} + R$$

$$\text{(iii)}\quad \kappa_{2r-1} = \bar{\kappa}_{2r-1} + R$$

$$\kappa_{2r} = \bar{\kappa}_{2r} - B_{2r}\frac{h^{2r}}{2r} + R$$

where B_j's are the Bernoulli numbers tabulated in table 17.9 and the Bernoulli polynomial $B_j^{(j+2)}(3/2)$ is equal to

$$\frac{(-1)^{j+1}(2j)!}{2^{2j}(j+1)!}\left(\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2j-1}\right) \text{for } j>1,$$

$B_0^{(2)}\left(\frac{3}{2}\right)=1$, $B_1^{(3)}\left(\frac{3}{2}\right)=0$. The remainder term R in each case is of the order $O(h^{2s})$.

Whenever the frequency curve $y=f(x)$ has a contact of high order at the extremities, conditions (a) and (b) are usually satisfied for moderate values of s and k. In such cases it has been found in practice that the result of applying the corrections is usually good even when h is not small. Putting $r=1, 2, \ldots$ and ignoring R we have the following **Sheppard's corrections** for the moments, factorial moments and cumulants.

mean and central moments	factorial moments	cumulants
$\mu'_1 = \bar{\mu}'_1$	$\mu'_{[1]} = \bar{\mu}'_{[1]}$	$\kappa_1 = \bar{\kappa}_1$
$\mu_2 = \bar{\mu}_2 - \frac{1}{12}h^2$	$\mu'_{[2]} = \bar{\mu}'_{[2]} - \frac{h^2}{12}$	$\kappa_2 = \bar{\kappa}_2 - \frac{h^2}{12}$
$\mu_3 = \bar{\mu}_3$	$\mu'_{[3]} = \bar{\mu}'_3 - \frac{h^2}{4}\bar{\mu}'_{[1]} + \frac{h^3}{4}$	$\kappa_3 = \bar{\kappa}_3$
$\mu_4 = \bar{\mu}_4 - \frac{1}{2}\bar{\mu}_2 h^2 + \frac{7}{240}h^4$	$\mu'_{[4]} = \bar{\mu}'_{[4]} - \frac{h^2}{2}\bar{\mu}'_{[2]} + h^3\bar{\mu}'_{[1]} - \frac{71}{80}h^4$	$\kappa_4 = \bar{\kappa}_4 + \frac{h^4}{120}$
$\mu_5 = \bar{\mu}_5 - \frac{5}{6}\bar{\mu}_3 h^2$		$\kappa_5 = \bar{\kappa}_5$
$\mu_6 = \bar{\mu}_6 - \frac{5}{4}\bar{\mu}_4 h^2 + \frac{7}{16}\bar{\mu}_2 h^4 - \frac{31}{1344}h^6$		$\kappa_6 = \bar{\kappa}_6 - \frac{h^6}{252}$

II. DISCRETE DISTRIBUTIONS

The tables of discrete distributions give the mean and variance in addition to the mgf, $M(t)$. Higher raw moments can be obtained by differentiating the mgf. Thus $\mu'_r = \left.\frac{d^rM}{dt^r}\right|_{t=0}$. Cumulants are obtained by differentiating the cgf, $K(t) = \log_e M(t)$. Thus $\kappa_r = \left.\frac{d^rK}{dt^r}\right|_{t=0}$. The pgf, $P(t) = M(\log_e t)$. The probability of x is $\frac{1}{x!}\left.\frac{d^xP}{dt^x}\right|_{t=0}$.

a. Basic distributions

INDIVIDUAL TERM, MEAN, VARIANCE AND MOMENT GENERATING FUNCTION

distribution notation	individual term probability of x	range of parameter	range of variable	mean	variance	moment generating function
Binomial $b(n, \pi)$	$\binom{n}{x}\pi^x(1-\pi)^{n-x}$	$0 < \pi < 1$	$0(1)n$	$n\pi$	$n\pi(1-\pi)$	$[(1-\pi)+\pi e^t]^n$
Poisson $p(\lambda)$	$e^{-\lambda}\lambda^x/x!$	$0 < \lambda < \infty$	$0(1)\infty$	λ	λ	$e^{\lambda(e^t-1)}$
Hypergeometric $h(N, N\pi, n)$	$\dfrac{\binom{N\pi}{x}\binom{N-N\pi}{n-x}}{\binom{N}{x}}$	$0 < \pi < 1$	$a(1)b$*	$n\pi$	$\frac{N-n}{N-1}[n\pi(1-\pi)]$	$\frac{(N-N\pi)^{[n]}}{N^{[n]}}\,{}_2F_1(-n, -N\pi; N-N\pi-n+1, e^t)$†
Negative binomial $n(\kappa, \rho)$	$\dfrac{\binom{\kappa+x-1}{x}\rho^x}{(1+\rho)^{\kappa+x}}$	$1 \leqslant \kappa$, $0 < \rho$	$0(1)\infty$	$\kappa\rho$	$\kappa\rho(1+\rho)$	$[1+\rho-\rho e^t]^{-\kappa}$
Logarithmic series $l(\pi)$	$-\alpha\pi^x/x$, $\alpha = 1/\log(1-\pi)$	$0 < \pi < 1$	$1(1)\infty$	$\frac{-\alpha\pi}{1-\pi}$	$\frac{-\alpha\pi(1+\alpha\pi)}{(1-\pi)^2}$	$\alpha\log(1-\pi e^t)$

*$b = \min\ (n, N\pi)$
$a = \max\ (0, n-N+N\pi)$

†${}_2F_1(a, b; c, x) = 1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}+\ldots$

b. Random sum distributions

A random sum distribution is the distribution of the sum of a random number n of independent identically distributed random variables. $p^+b(\lambda; k, p)$, to be read as Poisson sum of binomial, denotes the distribution of the sum of n independent binomial variables $b(k, p)$ with n as a random observation on the Poisson variable $p(\lambda)$. By convention, the sum assumes the value 0 whenever n is 0.

Let $P(t)$ be the probability generating function (pgf) of the random variable n and $M(t)$ be the moment generating function (mgf) of the distribution from which n observations are drawn. Then the mgf of the random sum distribution is $P(M(t))$.

distribution, range of parameter	individual term probability of x	range of variable	mean	variance	moment generating function
$b^+b(N, \pi; k, p)$ $0 < \pi < 1$ $0 < p < 1$	$\sum_{n=a_x}^{N} \binom{N}{n}\binom{nk}{x} \pi^n(1-\pi)^{N-n}p^x(1-p)^{nk-x}$	$0(1)Nk$	$N\pi kp$	$N\pi kp[q+(1-\pi)kp]$	$[(1-\pi)+\pi(q+pe^t)^k]^N$
$p^+b(\lambda; k, p)$ $0 < \lambda < \infty$ $0 < p < 1$	$\sum_{n=a_x}^{\infty} e^{-\lambda}\lambda^n \binom{nk}{x} p^x(1-p)^{nk-x}$	$0(1)\infty$	λkp	$\lambda kp[q+kp]$	$e^{\lambda[(q+pe^t)^k-1]}$
$n^+b(\kappa, \rho; k, p)$ $1 \leqslant \kappa$ $0 < \rho$ $0 < p < 1$	$\sum_{n=a_x}^{\infty} \binom{\kappa+n-1}{n} \frac{\rho^n}{(1+\rho)^{\kappa+n}} \binom{nk}{x} p^x(1-p)^{nk-x}$	$0(1)\infty$	$\kappa\rho kp$	$\kappa\rho kp[q+(1+\rho)kp]$	$[1+\rho-\rho(q+pe^t)^k]^{-\kappa}$
$l^+b(\pi; k, p)$ $0 < \pi < 1$ $0 < p < 1$	$\sum_{n=a_x}^{\infty} \frac{-\alpha\pi^n}{n} \binom{nk}{x} p^x(1-p)^{nk-x}$ $\alpha = 1/\log(1-\pi)$	$0(1)\infty$	$\frac{-\alpha\pi kp}{1-\pi}$	$\frac{-\alpha\pi kp}{1-\pi}\left[q+\frac{(1+\alpha\pi)kp}{1-\pi}\right]$	$\alpha \log [1-\pi(q+pe^t)^k]$

Note : (1) In the table $a_x = \left[\frac{x+k-1}{k}\right]$, *ie.* the greatest integer in $\frac{x+k-1}{k}$ and $q = 1-p$.

(2) Observe the special cases
(i) $b^+b(N, \pi; 1, p) = b(N, \pi p)$, (ii) $p^+b(\lambda; 1, p) = p(\lambda p)$, (iii) $n^+b(\kappa, \rho; 1, p) = n(\kappa, \rho p)$.

Random sum distributions (*continued*)

distribution, range of parameter	individual term probability of x	range of variable	mean	variance	moment generating function
$b{+}p(N, \pi; m)$ $0 < \pi < 1$ $0 < m$	$[\pi e^{-m}+(1-\pi)]^N$ if $x=0$ $\sum_{n=1}^{N}\binom{N}{n}\pi^n(1-\pi)^{N-n}\,e^{-nm}\frac{(nm)^x}{x!},\ x\neq 0$	$0(1)\infty$	$N\pi m$	$N\pi m[1+(1-\pi)m]$	$[(1-\pi)+\pi e^{m(e^t-1)}]^N$
$p{+}p(\lambda; m)$ $0 < \lambda$ $0 < m$	$e^{\lambda(e^{-m}-1)}$ if $x=0$ $\sum_{n=1}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}e^{-nm}\frac{(nm)^x}{x!},\ x\neq 0$	$0(1)\infty$	λm	$\lambda m[1+m]$	$\exp\,[\lambda(e^{m(e^t-1)}-1)]$
Th $(\lambda; m)$* $0 < \lambda$	$e^{-\lambda}$ if $x=0$ $\sum_{n=1}^{x}\frac{e^{-\lambda}\lambda^n e^{-nm}(nm)^{x-n}}{n!(x-n)!},\ x\neq 0$	$0(1)\infty$	$\lambda(1+m)$	$\lambda(1+3m+m^2)$	$\exp\,[\lambda(e^{me^t-m+t}-1)]$
$n{+}p(\kappa, \rho; m)$ $1 < k$ $0 < \rho$ $0 < m$	$[1+\rho-\rho e^{-m}]^{-\kappa}$ if $x=0$ $\sum_{n=1}^{\infty}\binom{\kappa+n-1}{n}\frac{\rho^n}{(1+\rho)^{\kappa+n}}e^{-nm}\frac{(nm)^x}{x!},\ x\neq 0$	$0(1)\infty$	$\kappa\rho m$	$\kappa\rho m[1+(1+\rho)m]$	$[1+\rho-\rho e^{m(e^t-1)}]^{-\kappa}$
$l{+}p(\pi; m)$ $0 < \pi < 1$ $0 < m$	$-\sum_{n=1}^{\infty}\alpha\frac{\pi^n}{n}e^{-nm}\frac{(nm)^x}{x!}$ $\alpha = 1/\log(1-\pi)$	$0(1)\infty$	$\frac{-\alpha\pi m}{1-\pi}$	$\frac{-\alpha\pi m}{1-\pi}\left[1+\frac{m(1+\alpha\pi)}{1-\pi}\right]$	$\alpha\log[1-\pi e^{m(e^t-1)}]$
$b{+}n(N, \pi; k, r)$ $0 < \pi < 1$ $0 < r,\ 1 \leqslant k$	$[(1-\pi)+\pi(1+r)^{-k}]^N$ if $x=0$ $\sum_{n=1}^{N}\binom{N}{n}\pi^n(1-\pi)^{N-n}\binom{nk+x-1}{n}\frac{r^x}{(1+r)^{nk+x}},\ x\neq 0$	$0(1)\infty$	$N\pi kr$	$N\pi kr[1+r+(1-\pi)kr]$	$[1-\pi+\pi(1+r-re^t)^{-k}]^N$

*Thomas distribution gives the distribution of a Poisson $[p(\lambda)]$ sum of independent identically distributed random variables X_i where X_i-1 has a Poisson $[p(m)]$ distribution.

Random sum distributions (*continued*)

distribution range of parameter	individual term probability of x	range of variable	mean	variance	moment generating function
$p^{+}n(\lambda\,;\,k, r)$ $0 < \lambda,$ $0 < r, 1 \leqslant k$	$e^{\lambda[(1+r)^{-k}-1]}$ if $x=0$ $\sum_{n=1}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\binom{nk+x-1}{x}\frac{r^x}{(1+r)^{nk+x}},\ x \neq 0$	$0(1)\infty$	λkr	$\lambda kr[1+r+kr]$	$\exp\{\lambda[(1+r-re^t)^{-k}-1]\}$
$n^{+}n(\kappa, \rho\,;\,k, r)$ $0 < \rho, r$ $1 \leqslant \kappa, k$	$[1+\rho-\rho(1+r)^{-k}]^{-\kappa},\ x=0$ $\sum_{n=1}^{\infty}\binom{\kappa+n-1}{n}\frac{\rho^n}{(1+\rho)^{\kappa+n}}\binom{nk+x-1}{x}\frac{r^x}{(1+r)^{nk+x}}$ $x \neq 0$	$0(1)\infty$	$\kappa\rho kr$	$\kappa\rho kr[1+r+(1+\rho)kr]$	$[1+\rho-\rho(1+r-re^t)^{-k}]^{-\kappa}$
$l^{+}n(\pi; k, r)$ $0 < \pi < 1$ $0 < r$ $1 \leqslant k$	$-\sum_{n=1}^{\infty}\frac{\alpha\pi^n}{n}\binom{nk+x-1}{x}\frac{r^x}{(1+r)^{nk+x}}$	$0(1)\infty$	$\frac{-\alpha\pi kr}{1-\pi}$	$\frac{-\alpha\pi kr}{1-\pi}\left[1+r+\frac{1+\alpha\pi}{1-\pi}kr\right]$	$\alpha\log[1-\pi(1+r-re^t)^{-k}]$
$b^{+}l(N, \pi\,;\,p)$ $0 < \pi, p < 1$	$\left.\frac{d^x[1-\pi+\pi a\log(1-pt)]^N}{x!\,dt^x}\right\|_{t=0}$	$0(1)\infty$	$\frac{-N\pi ap}{(1-p)}$	$\frac{-N\pi ap}{(1-p)^2}[1+\pi ap]$	$[1-\pi+\pi a\log(1-pe^t)]^N$
$p^{+}l(\lambda\,;\,p)$	$n(x\|\kappa, \rho)$ where $\kappa = -\lambda/\log(1-p)$ $\rho = p/(1-p)$	$0(1)\infty$	*	*	*
$n^{+}l(\kappa, \rho\,;\,p)$ $1 \leqslant \kappa$ $0 < \rho$ $0 < p < 1$	$\left.\frac{d^x[1+\rho-\rho a\log(1-pt)]^{-\kappa}}{x!\,dt^x}\right\|_{t=0}$	$0(1)\infty$	$\frac{-\kappa\rho ap}{1-p}$	$\frac{-\kappa\rho ap}{(1-p)^2}[1-\rho ap]$	$[1+\rho-\rho a\log(1-pe^t)]^{-\kappa}$
$l^{+}l(\pi\,;\,p)$ $0 < \pi, p < 1$	$\left.\frac{\alpha d^x\log[1-\pi a\log(1-pt)]}{x!\,dt^x}\right\|_{t=0}$	$1(1)\infty$	$\frac{\alpha\pi ap}{(1-\pi)(1-p)}$	$\frac{\alpha\pi ap}{(1-\pi)^2(1-p)^2}[1-\pi-\pi ap-\alpha\pi ap]$	$\alpha\log[1-\pi a\log(1-pe^t)]$

$\alpha = 1/\log(1-\pi)$, $a = 1/\log(1-p)$

*see the table of basic distributions (**IIa**)

c. Compound distributions

A compound distribution is formed by considering the parameter of a basic distribution as stochastic and obtaining the total probability of x by summing or integrating over the distribution of the parameter.

basic distribution	distribution of parameter	compound distribution, notation	individual term, range of parameter and of variable	mean	variance	moment generating function
$b(n, \pi)$	$n \sim b(N, \pi')$	$b(N, \pi\pi')$	*	*	*	*
$b(n, \pi)$	$n \sim p(\lambda)$	$p(\lambda \pi)$	*	*	*	*
$b(n, \pi)$	$n \sim n(\kappa, \rho)$	$n(\kappa, \pi\rho)$	*	*	*	*
$b(n, \pi)$	$n \sim l(\pi')$	$bl(\pi_1, \pi_2)$ $\pi_1 - 1 = \dfrac{\log(1-\pi'+\pi\pi')}{\log(1-\pi')}$ $\pi_2 = \dfrac{\pi\pi'}{1-\pi'+\pi\pi'}$	$1-\pi_1$ if $x = 0$ $-\pi_1\alpha_2\pi_2^x/x$ if $x \neq 0$ $\alpha_2 = 1/\log(1-\pi)$ $0 < \pi_1, \pi_2 < 1$ $x = 0(1)\infty$	$\dfrac{-\alpha_2\pi_1\pi_2}{1-\pi_2}$	$\dfrac{-\alpha_2\pi_1\pi_2(1+\alpha_2\pi_1\pi_2)}{(1-\pi_2)^2}$	$1-\pi_1+\pi_1\alpha_2 \log(1-\pi_2 e^t)$
$b(n, \pi)$	$\pi \sim B(\alpha, \beta)^{(1)}$	$bB(n, \alpha, \beta)$	$\dbinom{n}{x}\dfrac{B(\alpha+x, n+\beta-x)}{B(\alpha, \beta)}$ $0 < \alpha, \beta$ $x = 0(1)n$	$\dfrac{n\alpha}{\alpha+\beta}$	$\dfrac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$	—
$p(\lambda)$	$\dfrac{\lambda}{c} \sim p(\theta)$	$pp(\theta, c)$ (Neyman's contagious distribution—Type A)	$\dfrac{c^x e^{-\theta}}{x!}\sum_{k=0}^{\infty}\dfrac{k^x}{k!}(\theta e^{-c})^k$ $0 < \theta, c$ $x = 0(1)\infty$	$c\theta$	$c\theta(1+c)$	$\exp \theta[e^{c(e^t-1)}-1]$

Compound distributions (*continued*)

basic distribution	distribution of parameter	compound distribution notation	individual term, range of variable and of parameter	mean	variance	moment generating function
$p(\lambda)$	$\frac{\lambda}{c} \sim n(\kappa, \rho)$	$pn(\kappa, \rho, c)$	$\frac{c^x}{x!} \sum_{i=1}^{\infty} i^x e^{-ci} \binom{\kappa+i-1}{i} \frac{\rho^i}{(1+\rho)^{\kappa+i}}$ $1 \leqslant \kappa,\ 0 < \rho, c,\ x = 0(1)\infty$	$c\kappa\rho$	$c\kappa\rho$ $+c^2\kappa\rho(1+\rho)$	$[1+\rho-\rho e^{c(e^t-1)}]^{-\kappa}$
$p(\lambda)$	$\lambda \sim G(r, \theta)^{(1)}$	$n(r, \theta)$	*	*	*	*
$P(k, \pi)$ Pascal(2)	$k \sim p(\lambda)$	$Pp(\lambda, \pi)$ (Polya Aeppli)	$e^{-\lambda}$ if $x = 0$ $\pi^x e^{-\lambda} \sum_{j=1}^{x} \binom{x-1}{j-1} \frac{1}{j!} \left[\frac{\lambda(1-\pi)}{\pi}\right]^j,\ x \neq 0$ $0 < \lambda,\ 0 < \pi < 1,\ x = 0\,(1)\,\infty$	$\frac{\lambda}{1-\pi}$	$\frac{\lambda(\pi+1)}{(1-\pi)^2}$	$e^{\lambda(e^t-1)/(1-\pi e^t)}$

* see the table of basic distributions (**IIa**)

(1) $B(\alpha, \beta)$ and $G(r, \theta)$ refer respectively to the Beta and Gamma distributions, described in **III** (continuous distributions).

(2) The frequency function of the Pascal distribution is given by $P(x|k, \pi) = n(x-k|\kappa, \rho)$ for $x = k(1)\infty$ where the parameters of the negative binomial are $\rho = \frac{\pi}{1-\pi}$ and $\kappa = k$ an integer.

d. Distribution functions of some discrete distributions

Binomial $b(n, \pi)$:
$$\sum_{x=0}^{s} \binom{n}{x} \pi^x (1-\pi)^{n-x} = \frac{1}{B(n-s, s+1)} \int_0^{1-\pi} y^{n-s-1}(1-y)^s\, dy.$$

Poisson $p(\lambda)$:
$$\sum_{x=0}^{s} e^{-\lambda} \frac{\lambda^x}{x!} = \frac{1}{\Gamma(s+1)} \int_{\lambda}^{\infty} e^{-y} y^s\, dy.$$

Negative binomial $n(\kappa, \rho)$:
$$\sum_{x=0}^{s} \binom{\kappa+x-1}{x} \frac{\rho^x}{(1+\rho)^{\kappa+x}} = \frac{1}{B(s+1, \kappa)} \int_{\rho}^{\infty} y^s (1+y)^{-(\kappa+s+1)}\, dy.$$

III. CONTINUOUS DISTRIBUTIONS

a. Basic distributions

DENSITY FUNCTION, MEAN, VARIANCE AND CHARACTERISTIC FUNCTION

distribution and notation	density function	range of parameter	range of variable	mean	variance	characteristic function
Normal $N(\mu, \sigma)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$	$-\infty < \mu < \infty$ $0 < \sigma < \infty$	$-\infty < x < \infty$	μ	σ^2	$e^{i\mu t - \sigma^2 t^2/2}$
Truncated Normal $N_a^b(\mu, \sigma)$	$N(x\|\mu, \sigma)/P$ $\left[P = \int_a^b N(x\|\mu, \sigma)dx\right]$	$-\infty < \mu < \infty$ $0 < \sigma < \infty$	$a \leqslant x \leqslant b$	$\mu+\sigma\theta$ $a' = (a-\mu)/\sigma$, $b' = (b-\mu)/\sigma$ $\theta = [N(a')-N(b')]/P$	$\sigma^2\left[\frac{P+a'N(a')-b'N(b')}{P}-\theta^2\right]$	—
Log Normal $LN(\lambda, \rho)$	$\frac{\delta}{x\sqrt{2\pi}} e^{-\frac{1}{2}(\gamma+\delta \log x)^2}$	$-\infty < \gamma < \infty$ $0 < \delta < \infty$	$0 \leqslant x < \infty$	$\omega\rho$ $\rho = e^{-\gamma/\delta}$ $\omega = e^{1/2\delta^2}$	$\omega^2\rho^2(\omega^2-1)$	—
Cauchy $C(\mu, \lambda)$	$\frac{\lambda}{\pi[\lambda^2+(x-\mu)^2]}$	$-\infty < \mu < \infty$ $0 < \lambda < \infty$	$-\infty < x < \infty$	*	*	$e^{i\mu t - \|t\lambda\|}$
Rectangular $R(a, b)$	$\frac{1}{b-a}$	$-\infty < a < b < \infty$	$a \leqslant x \leqslant b$	$(a+b)/2$	$(b-a)^2/12$	$(e^{itb}-e^{ita})/it(b-a)$
Exponential $Exp\ (\theta)$	$\theta e^{-\theta x}$	$0 < \theta < \infty$	$0 \leqslant x \leqslant \infty$	$1/\theta$	$1/\theta^2$	$\frac{\theta}{\theta-it}$

Note: If $f(\theta)$ is the notation for a distribution with parameter θ, the density at x will be denoted by $f(x|\theta)$. Thus $N(x|\mu, \sigma)$ denotes the density of the normal distribution $N(\mu, \sigma)$

Basic distributions (*continued*)

distribution and notation	density function	range of parameter	range of variable	mean	variance	characteristic function
Gamma $G(r, \theta)$	$\dfrac{\theta^r}{\Gamma(r)} x^{r-1} e^{-\theta x}$	$0 < r < \infty$	$0 \leqslant x < \infty$	r/θ	r/θ^2	$\left(\dfrac{\theta}{\theta - it}\right)^r$
Chisquare $\chi^2(\nu)$	$\dfrac{e^{-x/2}(x)^{\frac{\nu-2}{2}}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}$	$\nu = 1(1)\infty$	$0 \leqslant x < \infty$	ν	2ν	$\dfrac{1}{(1-2it)^{\frac{\nu}{2}}}$
Student's t $t(\nu)$	$\dfrac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)}\left(1+\dfrac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$	$\nu = 1(1)\infty$	$-\infty < x < \infty$	0 for $\nu \geqslant 2$	$\nu/(\nu-2)$ for $\nu \geqslant 3$	—
Beta $B(m, n)$	$\dfrac{1}{B(m, n)} x^{m-1}(1-x)^{n-1}$	$0 < m < \infty$, $0 < n < \infty$	$0 \leqslant x \leqslant 1$	$m/(m+n)$	$mn/(m+n)^2(m+n+1)$	—
Fisher's F $F(\nu_1, \nu_2)$	$\dfrac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1-2}{2}}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left(1+\frac{\nu_1}{\nu_2}x\right)^{\frac{\nu_1+\nu_2}{2}}}$	$\nu_1 = 1(1)\infty$, $\nu_2 = 1(1)\infty$	$0 \leqslant x < \infty$	$\nu_2/(\nu_2-2)$ for $\nu_2 \geqslant 3$	$\dfrac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$ for $\nu_2 \geqslant 5$	—
Laplace $L(\mu, \sigma)$	$\dfrac{1}{\sigma\sqrt{2\pi}} e^{-\sqrt{2}\lvert x-\mu\rvert/\sigma}$	$-\infty < \mu < \infty$, $0 < \sigma < \infty$	$-\infty < x < \infty$	μ	σ^2	$e^{it\mu}(1+\sigma^2t^2)^{-1}$

b. Some non-central distributions (density functions)

(i) *Bivariate normal* (with means μ_1, μ_2; variances σ_1^2, σ_2^2 and correlation ρ)

$$N_2(x_1, x_2 \mid \mu_1, \mu_2; \sigma_1, \sigma_2, \rho)$$

$$= (2\pi\sigma_1\sigma_2\sqrt{1-\rho^2})^{-1}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}\right]$$

(ii) *Multivariate normal* (with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$)

$$N_p(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \quad (2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left[-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

where $\boldsymbol{\Sigma}^{-1}$ is the inverse of $\boldsymbol{\Sigma}$.

(iii) *Wishart distribution*

$$W_p(\mathbf{S} \mid \nu, \boldsymbol{\Sigma}) = \left[2^{p/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{\nu-i+1}{2}\right)\right]^{-1}$$

$$\times|\boldsymbol{\Sigma}|^{-1/2}|S|^{(\nu-p-1)/2}e^{-(\mathrm{tr}\,\boldsymbol{\Sigma}^{-1}\mathbf{S})/2}$$

If $\mathbf{X}_1, \mathbf{X}_2 \ldots, \mathbf{X}_\nu$ are independent $N_p(0, \boldsymbol{\Sigma})$ variables then $\mathbf{S} = \sum_1^{\nu}\mathbf{X}_i'\mathbf{X}$ has the above distribution.

(iv) *Noncentral* χ^2

$$\chi^2(x \mid \nu, \lambda) = e^{-\frac{\lambda}{2}}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\lambda}{2}\right)^r G\left(x \mid \frac{1}{2}, \frac{\nu}{2}+r\right), \; 0 < x < \infty$$

where ν is called degrees of freedom, λ the noncentrality parameter, and

$$G(x \mid \alpha, p) = \alpha^p[\Gamma(p)]^{-1}e^{-x\alpha}x^{p-1}.$$

Note: The distribution of Σx_i^2, where x_i is distributed as $N_1(\mu_i, \sigma^2)$ and x_i are all independent, is non-central $\chi^2(\nu, (\Sigma\mu_i^2)/\sigma^2)$.

(v) *Noncentral* t

$$t(x \mid \nu, \delta) = \frac{\nu^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)}\frac{e^{-\delta^2/2}}{(\nu+x^2)^{(\nu+1)/2}}\sum_{r=0}^{\infty}\Gamma\left(\frac{\nu+r+1}{2}\right)\left(\frac{\delta^r}{r!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{r/2}$$

with $-\infty < x < \infty$, where ν is the degrees of freedom and δ is the noncentrality parameter.

Note: The distribution of $X/\sqrt{(Y/\nu)}$ where X and Y are independently distributed, X as $N(\delta, 1)$ and Y as $\chi^2(\nu)$ variates, is noncentral $t(\nu, \delta)$.

(vi) *Noncentral* F

$$F(x \mid \nu_1, \nu_2, \lambda) = e^{-\lambda^2/2}\frac{\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}x^{\frac{\nu_1}{2}-1}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left(1+\frac{\nu_1}{\nu_2}x\right)^{\frac{\nu_1+\nu_2}{2}}}\,{}_1F_1\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2}, \frac{\lambda^2\nu_1 x}{2(\nu_2+\nu_1 x)}\right)$$

with $0 \leqslant x < \infty$, where ${}_1F_1$ is the hypergeometric function of the first kind defined by

$$ {}_1F_1(a, b, y) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+r)} \frac{y^r}{r!}. $$

Note : The distribution of $(X/\nu_1)/(Y/\nu_2)$, where X and Y are independently distributed, X being non-central $\chi^2(\nu_1, \lambda)$ and Y a central $\chi^2(\nu_2)$, is noncentral $F(\nu_1, \nu_2, \lambda)$.

(vii) *Multiple correlation*

Let R^2 be the square of the multiple correlation, based on a sample of size n, of one variable on $p-1$ other variables. If the latter are considered fixed, then the density function of R^2 is

$$ R^2(x \mid p, n, \delta) = e^{-\delta^2/2} B\left(x \Big| \frac{p-1}{2}, \frac{n-p}{2}\right) {}_1F_1\left(\frac{n-1}{2}, \frac{p-1}{2}, \frac{\delta^2 R^2}{2}\right) $$

with $0 \leqslant x < \infty$, which is called multiple correlation distribution of the first kind. If variations in the $(p-1)$ variables are allowed, then the density function is

$$ R^2(x \mid p, n, \rho) = (1-\rho^2)^{(n-1)/2} B\left(x \Big| \frac{p-1}{2}, \frac{n-p}{2}\right) {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}, \frac{p-1}{2}, \rho^2 x\right) $$

with $0 \leqslant x \leqslant 1$, where ρ is the population multiple correlation coefficient and ${}_2F_1$ is the hypergeometric function of the second kind defined by

$$ {}_2F_1(a, b, c, y) = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{\Gamma(a)} \frac{\Gamma(b+r)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+r)} \frac{y^r}{r!}, $$

and $$ B(x \mid a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} $$

(viii) *Hotelling's T^2, Mahalanobis' D^2*

$$ T^2(x \mid p, \nu, c\,\tau^2) = \frac{\nu-p+1}{p} F\left(\frac{\nu-p+1}{p} x \mid p, \nu-p+1, c\tau^2\right) $$

where the function F is that of non-central F distribution.

Note : The distribution of $\mathbf{d}'\ \mathbf{S}^{-1}\ \mathbf{d}$ has the above form if $\mathbf{d}$ has the p-variate normal distribution $N_p(\boldsymbol{\delta}, \mathrm{c}^{-1}\,\boldsymbol{\Sigma})$, c a scalar, and the elements of the matrix $\mathbf{S}$ have an independent Wishart distribution $W_p(\mathbf{S}, \boldsymbol{\Sigma})$. In such a case $\tau^2 = \boldsymbol{\delta}\,\boldsymbol{\Sigma}^{-1}\,\boldsymbol{\delta}'$.

IV STANDARD ERRORS

a. Application

In large sample theory, hypotheses concerning unknown population parameters, can be tested in a simple way by using efficient estimators and their standard errors.

Thus if T is an estimator of θ based on n observations and $\sigma(\theta)/\sqrt{n}$ is the standard error of T, then to test the hypothesis $H_0 : \theta = \theta^0$, $\sqrt{n}(T-\theta^0)/\sigma(\theta^0)$ will be used as a standard normal deviate.

To test whether two parallel independent estimators T_1 and T_2 with standard errors $\sigma_1(\theta)/\sqrt{n_1}$ and $\sigma_2(\theta)/\sqrt{n_2}$ are in agreement (i.e., whether they estimate the same parametric value) the appropriate standard normal deviate will be

$$(T_1-T_2) \div \sqrt{\frac{\sigma_1^2(T_1)}{n_1}+\frac{\sigma_2^2(T_2)}{n_2}}$$

where in the expressions for standard errors, estimates are substituted for unknown parameters provided $\sigma_1(\theta)$ and $\sigma_2(\theta)$ are continuous functions of θ.

To test whether k parallel independent estimators $T_1, T_2, \ldots, T_k$ having standard errors $\sigma_1(\theta)/\sqrt{n_1}$, $\sigma_2(\theta)/\sqrt{n_2}$, ..., $\sigma_k(\theta)/\sqrt{n_k}$, are in agreement, the test-statistic

$$H = \sum_{i=1}^{k} \frac{n_i}{\sigma_i^2(T_i)} (T_i-\overline{T})^2$$

may be used as a chi-square with $k-1$ d.f., where

$$\overline{T} = \sum_{i=1}^{k} \frac{n_i T_i}{\sigma_i^2(T_i)} \Big/ \sum_{i=1}^{k} \frac{n_i}{\sigma_i^2(T_i)}.$$

b. Standard errors of some statistics

Notations for population parameters :

$\mu_1 =$ mean, $\quad \mu_r = r$-th central moment

$\beta_{2n} = \mu_{2n+2}/\mu_2^{n+1}, \quad \beta_{2n+1} = \mu_3\mu_{2n+3}/\mu_2^{n+3}$

$\mu_{rs} = (r, s)$-th bivariate central moment

$\rho =$ correlation coefficient

$\theta =$ percentage coefficient of variation

$\delta =$ mean deviation about the mean

$\Delta =$ Gini's mean difference

$\phi = E|X-Y|\,|X-Z|$ where X, Y, Z are three independent observations drawn from the same population

$=$ ordinate at the p-th quantile.

The standard errors of some of the more common statistics are given below

STANDARD ERRORS OF SOME STATISTICS

(To obtain the standard error, divide the tabular entry by n, the sample size and take the square root)

statistic	arbitrary distribution	normal distribution
mean $\bar{x}$	$\mu_2\ (=\sigma^2)$	σ^2
variance s^2	$\mu_4-\mu_2^2$	$2\sigma^4$
standard deviation s	$(\mu_4-\mu_2^2)/4\mu_2$	$\sigma^2/2$
k-th central moment m_k	$\mu_{2k}-\mu_k^2+k^2\mu_2\mu_{k-1}^2-2k\mu_{k-1}\mu_{k+1}$	—
coefficient of variation (%)	$\theta^2\left[\frac{\mu_4-\mu_2^2}{4\mu_2^2}+\frac{\mu_2}{\mu_1'^2}-\frac{\mu_3}{\mu_2\mu_1'}\right]$	$\frac{\theta^2}{2}\left[1+2\left(\frac{\theta}{100}\right)^2\right]$
sample $\sqrt{\beta_1}$	$[4\beta_4-24\beta_2+36+9\beta_1\beta_2-12\beta_3+35\beta_1]/4$	6
sample β_2	$\beta_6-4\beta_2\beta_4+4\beta_2^3-\beta_2^2+16\beta_2\beta_1-8\beta_3+16\beta_1$	24
mean deviation about sample mean	$\sigma^2-\delta^2$	$\sigma^2(1-2/\pi)$
Gini's mean difference	$4(\phi-\Delta^2)$	$(0.8068)^2\sigma^2$
median $\tilde{x}$	$1/4y_{0\cdot 5}^2$	$(1.2533)^2\sigma^2$
quartile	$3/4y^2(y=y_{0\cdot 25}$ or $y_{0\cdot 75})$	$(1.3626)^2\sigma^2$
p-th quantile*	$p(1-p)/y_p^2$	—
semi-interquartile range	$\frac{1}{4}\left[\frac{3}{16}\left(y_{0.25}^{-2}+y_{0.75}^{-2}\right)-\frac{1}{8}y_{0.25}^{-1}y_{0.75}^{-1}\right]$	$(0.7867)^2\sigma^2$
correlation coefficient	$\rho^2\left[\frac{\mu_{22}}{\mu_{11}^2}+\frac{1}{4}\left(\frac{\mu_{40}}{\mu_{20}^2}+\frac{\mu_{04}}{\mu_{02}^2}+\frac{2\mu_{22}}{\mu_{20}\mu_{02}}\right)-\left(\frac{\mu_{31}}{\mu_{11}\mu_{20}}+\frac{\mu_{13}}{\mu_{11}\mu_{02}}\right)\right]$	$(1-\rho^2)^2$

* for a normal population the standard errors of the sample deciles are as follows:

4th and 6th deciles: $1.2680\sigma/\sqrt{n}$, 3rd and 7th deciles: $1.3180\sigma/\sqrt{n}$

2nd and 8th deciles: $1.4288\sigma/\sqrt{n}$, 1st and 9th deciles: $1.7094\sigma/\sqrt{n}$.

The asymptotic covariance between the p-th and p'-th quantiles ($p<p'$) is $pq'/ny_py_{p'}$ where $q'=1-p'$. Thus for a normal distribution the asymptotic covariance between the first quartile and the median is equal to $0.9860\sigma^2/n$.

c. Transformation of statistics

For the application of techniques such as the analysis of variance it may be necessary to use transformed value of an estimate so that the asymptotic variance (square of the standard error) is independent of the unknown parameter. Some standard transformations and the corresponding asymptotic variances appear in the table given in the next page

d. Normalisation of frequency functions

A large number of statistics tend to be normally distributed as sample size n increases. Let T be such a statistic with k_i as its i-th cumulant. Assume further the existence of constants μ and σ such that

$$\rho_1 = (k_1-\mu)/\sigma = O(n^{-\frac{1}{2}})$$

$$\rho_2 = (k_2-\sigma^2)/\sigma^2 = O(n^{-1})$$

and $$\rho_r = k_r/\sigma^r = O(n^{1-\frac{1}{2}r}) \qquad \text{for} \quad r = 3, 4, \ldots,.$$

Define $x = (T-\mu)/\sigma$. The following equation, gives to order n^{-2}, an expression for a transformed variable y which has standard normal distribution :

$$\begin{aligned}
x-y = {} & \rho_1+\frac{1}{6}\rho_3(x^2-1)+\frac{1}{2}\rho_2 x-\frac{1}{3}\rho_1\rho_3 x+\frac{1}{24}\rho_4(x^3-3x) \\
& -\frac{1}{36}\rho_3^2(4x^3-7x)-\frac{1}{2}\rho_1\rho_2+\frac{1}{6}\rho_1^2\rho_3-\frac{1}{12}\rho_2\rho_3(5x^2-3)-\frac{1}{8}\rho_1\rho_4(x^2-1) \\
& +\frac{1}{120}\rho_5(x^4-6x^2+3)+\frac{1}{36}\rho_1\rho_3^2(12x^2-7)-\frac{1}{144}\rho_3\rho_4(11x^4-42x^2+15) \\
& +\frac{1}{648}\rho_3^3(69x^4-187x^2+52)-\frac{3}{8}\rho_2^2 x+\frac{5}{6}\rho_1\rho_2\rho_3 x+\frac{1}{8}\rho_1^2\rho_4 x-\frac{1}{48}\rho_2\rho_4(7x^3-15x) \\
& -\frac{1}{30}\rho_1\rho_5(x^3-3x)+\frac{1}{720}\rho_6(x^5-10x^3+15x)-\frac{1}{3}\rho_1^2\rho_3^2 x+\frac{1}{72}\rho_2\rho_3^2(36x^3-49x) \\
& -\frac{1}{384}\rho_4^2(5x^5-32x^3+35x)+\frac{1}{36}\rho_1\rho_3\rho_4(11x^3-21x)-\frac{1}{360}\rho_3\rho_5(7x^5-48x^3+51x) \\
& -\frac{1}{324}\rho_1\rho_3^3(138x^3-187x)+\frac{1}{864}\rho_3^2\rho_4(111x^5-547x^3+456x) \\
& -\frac{1}{7776}\rho_3^4(948x^5-3628x^3+2473x).
\end{aligned}$$

The following equation connecting x with y is equally useful :

$$\begin{aligned}
x-y = {} & \rho_1+\frac{1}{6}\rho_3(y^2-1)+\frac{1}{2}\rho_2 y+\frac{1}{24}\rho_4(y^3-3y)-\frac{1}{36}\rho_3^2(2y^3-5y)-\frac{1}{6}\rho_2\rho_3(y^2-1) \\
& +\frac{1}{120}\rho_5(y^4-6y^2+3)-\frac{1}{24}\rho_3\rho_4(y^4-5y^2+2)+\frac{1}{324}\rho_3^3(12y^4-53y^2+17) \\
& -\frac{1}{8}\rho_2^2 y-\frac{1}{16}\rho_2\rho_4(y^3-3y)+\frac{1}{720}\rho_6(y^5-10y^3+15y)+\frac{1}{72}\rho_2\rho_3^2(10y^3-25y) \\
& -\frac{1}{384}\rho_4^2(3y^5-24y^3+29y)-\frac{1}{180}\rho_3\rho_5(2y^5-17y^3+21y) \\
& +\frac{1}{288}\rho_3^2\rho_4(14y^5-103y^3+107y)-\frac{1}{7776}\rho_3^4(252y^5-1688y^3+1511y).
\end{aligned}$$

Special cases of this formula corresponding to the t, χ^2 and F distributions are discussed in appropriate sections dealing with the different tables relating to these statistics.

(a)

SOME STANDARD TRANSFORMATIONS AND THEIR ASYMPTOTIC VARIANCES

population parameter	estimator	asymptotic variance	transformed value: parameter	transformed value: estimator	transformed value: asymptotic variance
binomial proportion π	sample proportion $p = r/n$	$\frac{\pi(1-\pi)}{n}$	$\sin^{-1}\sqrt{\pi}$	$\sin^{-1}\sqrt{p}$	$\frac{1}{4n}$
				$^*\sin^{-1}\sqrt{\frac{r+3/8}{n+3/4}}$	$\frac{1}{(4n+2)}$
Poisson mean λ	Poisson observation x	λ	$\sqrt{\lambda}$	$\sqrt{x}$	$\frac{1}{4}$
			$\sqrt{\lambda+3/8}$	$^*\sqrt{x+3/8}$	$\frac{1}{4}$
correlation coefficient ρ in bivariate normal	sample correlation coefficient r	$\frac{(1-\rho^2)^2}{n}$	$\tanh^{-1}\rho$	$\tanh^{-1}r$	$\frac{1}{(n-3)}$
intraclass correlation coefficient ρ					
(i) bivariate normal	sample intraclass correlation coefficient r	$\frac{(1-\rho^2)^2}{n}$	$\tanh^{-1}\rho$	$\tanh^{-1}r$	$\frac{1}{n-\frac{3}{2}}$
(ii) k-variate normal		$\frac{2(1-\rho)^2(1+\overline{k-1}\rho)^2}{k(k-1)n}$	$\frac{1}{2}\log_e\frac{1+(k-1)\rho}{1-\rho}$	$\frac{1}{2}\log_e\frac{1+(k-1)r}{1-r}$	$\frac{k}{2(k-1)(n-2)}$

* comparatively rapid stabilisation is achieved through this refinement due to Anscombe.

V. SAMPLE SURVEY ESTIMATES AND THEIR STANDARD ERRORS

a. Notations

The following notations[1, 2] are used for sample statistics and the corresponding population characteristics where y indicates the primary variate under investigation, x a supplementary variate and r the variable ratio y/x.

	sample	population
number of units	n	N
sampling fraction	$f = \frac{n}{N}$	—
raising factor	$g = \frac{1}{f}$	—
summation over constituent units	S	Σ
arithmetic mean of y, x, r	$\bar{y}, \bar{x}, \bar{r}$	μ_y, μ_x, μ_r
ratio of means	$\hat{\xi} = \frac{\bar{y}}{\bar{x}}$	$\xi = \frac{\mu_y}{\mu_x}$
variance of y[3]	$s_y^2 = \frac{1}{n-1} S(y-\bar{y})^2$	$\sigma_y^2 = \frac{1}{N} \Sigma(y-\mu_y)^2$
		$\sigma_y'^2 = \frac{1}{N-1} \Sigma(y-\mu_y)^2$
covariance of x and y	$s_{xy} = \frac{1}{n-1} S(x-\bar{x})(y-\bar{y})$	$\sigma_{xy} = \frac{1}{N} \Sigma(x-\mu_x)(y-\mu_y)$
		$\sigma'_{xy} = \frac{1}{N-1} \Sigma(x-\mu_x)(y-\mu_y)$
regression coefficient (y on x)	$b = \frac{s_{xy}}{s_x^2}$	$\beta = \frac{\sigma_{xy}}{\sigma_x^2}$

(1) A suffix i to these symbols will imply that the corresponding definition has to be understood in terms of the i-th stratum.

(2) A curl on top will represent sample estimate. Thus $\hat{\mu}_y$ represents the estimate of μ_y and $\hat{V}(\hat{\mu}_y)$ represents an estimate of $V(\hat{\mu}_y)$ the variance of $\hat{\mu}_y$.

(3) Sample (population) variance of x, r denoted by s_x^2, (σ_x^2, $\sigma_x'^2$) and s_r^2, (σ_r^2, $\sigma_r'^2$) are define in a similar manner.

b. Common methods of sampling, estimates, and standard errors

method of sampling	estimate	formula for variance of estimate[1,2]
simple random sampling	$\bar{y}$	$\frac{1-f}{n}\sigma_y'^2$
stratified simple random[3] sampling	$\Sigma\pi_i\bar{y}_i$ $(\pi_i = N_i/N)$	$\Sigma\pi_i^2\frac{(1-f_i)}{n_i}\sigma_{yi}'^2$

(1) The formula is given for 'without replacement' sampling. For sampling with replacement the formula is obtained by putting the corresponding $f = 0$ and dropping the prime (′) from the corresponding σ'^2.

(2) The expression for an estimate of this variance is obtained by substituting s_y^2 for $\sigma_y'^2$ (or σ_y^2) and s_{yi}^2 for $\sigma_{yi}'^2$ (or σ_{yi}^2) wherever necessary.

(3) For stratified sampling, in the general case, the formulae for estimate and its variance are

$$\hat{\mu}_y = \Sigma\pi_i\hat{\mu}_{yi},\ V(\hat{\mu}_y) = \Sigma\pi_i^2 V(\hat{\mu}_{yi})$$

where $\hat{\mu}_{yi}$ is the estimate for i-th stratum mean and $V(\hat{\mu}_{yi})$ is the variance of the estimate

c. Methods of estimation using supplementary variable

For simple random sampling, when μ_x is known, the formulae are as follows :

method of estimation	estimate	formula[1,2] for variance of estimate[3]
ratio method	$\frac{\bar{y}}{\bar{x}}\mu_x$	$\frac{1-f}{n}(\sigma_y'^2-2\xi\sigma_{xy}'+\xi^2\sigma_x'^2)$
product method	$\frac{\bar{y}\bar{x}}{\mu_x}$	$\frac{1-f}{n}(\sigma_y'^2+2\xi\sigma_{xy}'+\xi^2\sigma_x'^2)$
difference method	$(\bar{y}-\bar{x})+\mu_x$	$\frac{1-f}{n}(\sigma_y'^2-2\sigma_{xy}+\sigma_x'^2)$
regression method[4]	$\bar{y}+b(\mu_x-\bar{x})$	$\frac{1-f}{n}(\sigma_y'^2-\beta^2\sigma_x'^2)$

(1) The formula for variance is approximate except for the difference method. The approximation assumes that the sample size is large.

(2) The formula is given for 'without replacement' sampling. For sampling with replacement the formula is obtained by putting $f = 0$ and by dropping the prime (′) in σ_x, σ_y and σ_{xy}.

(3) The expression for an estimate of the variance is obtained by substituting s_x^2 for $\sigma_x'^2$ (or σ_x^2), s_y^2 for $\sigma_y'^2$ (or σ_y^2), s_{xy} for σ_{xy}' (or σ_{xy}), $\hat{\xi}$ for ξ and b for β wherever necessary.

(4) The regression coefficient b is estimated from the sample on (y, x) by the formula s_{xy}/s_x^2 (see the table of notations).

d. Modifications required for two-phase sampling

Consider the situation when μ_x is unknown and sampling for x is cheaper than sampling for y. In such cases, we take a sample of size n units for obtaining x and y and an independent and larger sample (of size n' and sampling fraction $f' > f$) covering the x's only. Then an estimate of μ_x is obtained from the second sample and substituted in the formula for estimates in section **c**. For such estimates of μ_y, expressions for variance would be as follows.

method of estimation	formula[1, 2] for variance of estimate[3] in two-phase sampling
ratio method	$\frac{1-f}{n}(\sigma_y'^2 - 2\xi\sigma_{xy}' + \xi^2\sigma_x'^2) + \frac{1-f'}{n'} \cdot \xi^2\sigma_x'^2$
product method	$\frac{1-f}{n}(\sigma_y'^2 + 2\xi\sigma_{xy}' + \xi^2\sigma_x'^2) + \frac{1-f'}{n'} \xi^2\sigma_x'^2$
difference method	$\frac{1-f}{n}(\sigma_y'^2 - 2\sigma_{xy}' + \sigma_x'^2) + \frac{1-f'}{n'} \sigma_x'^2$
regression method	$\frac{1-f}{n}(\sigma_y'^2 - \beta^2\sigma_x'^2) + \frac{1-f'}{n'} \beta^2\sigma_x'^2$

(1), (2) and (3). See footnote to table in section **c**.

e. Sampling with replacement and with probabilities proportional to size (x)

Estimate : $\hat{\mu}_y = \bar{r}\mu_x$

Variance of estimate : $V(\hat{\mu}_y) = \frac{\mu_x^2\sigma_r^2}{n}$

Estimate of variance : $\hat{V}(\hat{\mu}_y) = \frac{\mu_x^2 s_r^2}{n}$

f. Two-stage sampling schemes

A two-stage sampling scheme specifies m_1, the number of first stage units that will be selected in the sample out of a total of M_1 such units in the population and also m_{2i}, the number of second stage units (subunits) that will be included in the sample out of a total of M_{2i} subunits contained in the i-th first stage unit in case

this particular first stage unit is chosen through the first stage selection. Let

$$g_1 = \frac{1}{f_1} = \frac{M_1}{m_1}$$

$$g_{2i} = \frac{1}{f_{2i}} = \frac{M_{2i}}{m_{2i}}.$$

Note that though g_1 is necessarily a constant g_{2i} could possibly vary from one first stage unit to another. For the i-th first stage unit let the total, mean and variance of all the second stage units be denoted by τ_{yi}, μ_{yi} and σ^2_{yi} and if the i-th first stage unit is included in the sample, let the corresponding sample figures be denoted by T_{yi}, $\bar{y}_i$ and s^2_{yi}. If the first stage selection is based on simple random sampling, we have

Estimate : $$\hat{\mu}_y = \frac{g_1 \underset{i}{S} \hat{\tau}_{yi}}{N}$$

and

Variance : $$V(\hat{\mu}_y) = \frac{1-f_1}{m_1}(\sigma_1')^2 + \frac{g_1}{N^2}\sum_i V(\hat{\tau}_{yi})$$

where $N = \Sigma M_{2i}$, $(\sigma_1')^2 = \frac{M_1^2}{N^2}\left\{\frac{1}{M_1-1}\Sigma(\tau_{yi}-\bar{\tau}_y)^2\right\}$, $\bar{\tau}_y = \frac{1}{M_1}\Sigma\,\tau_{yi} = \frac{N}{M_1}\mu_y$. For example for a simple random sample of second stage unit $g_{2i}\,T_{yi}$ provides an unbiased estimate for τ_{yi} and $V(g_{2i}\;T_{yi}) = \frac{M_{2i}^2(1-f_2)}{m_{2i}}(\sigma_{yi}')^2$. In the special case where $M_{2i} = M_2$ and $m_{2i} = m_2$ $(i = 1, 2, \ldots, M_1)$ this estimate of τ_{yi} leads to the following estimate of μ_y

$$\hat{\mu}_y = \bar{\bar{y}} = \frac{1}{m_1}\underset{i}{S}\bar{y}_i$$

the grandmean of all the sample observations. We have

$$V(\bar{\bar{y}}) = \frac{1-f_1}{m_1}(\sigma_1')^2 + \frac{1-f_2}{m_1 m_2}(\sigma_2')^2$$

where

$$(\sigma_1')^2 = \frac{1}{M_1-1}\Sigma(\mu_{yi}-\mu_y)^2 \text{ and } (\sigma_2')^2 = \frac{1}{M_1}\sum_i(\sigma_{yi}')^2,$$

and

$$\hat{V}(\bar{\bar{y}}) = \frac{1-f_1}{m_1}s_1^2 + f_1\,\frac{1-f_2}{m_1 m_2}s_2^2$$

where

$$s_1^2 = \frac{1}{m_1-1}\underset{i}{S}(\bar{y}_i-\bar{\bar{y}})^2 \text{ and } s_2^2 = \frac{1}{m_1}\underset{i}{S}\,s_{yi}^2.$$

VI. NUMERICAL ANALYSIS

a. Interpolation

Interpolation is a process for determining approximately the value of a function $y = f(x)$ at an untabulated value x of the argument within the range of tabulation, on the basis of a given set of tabulated values of the function. In polynomial interpolation the knowledge of the tabulated values is used to estimate the function, the form of which may be unknown, by a polynomial of sufficiently high degree and the approximating polynomial is used to compute the required intermediate value. Some formulae for polynomial interpolation are given in this chapter. These are appropriate for tables in which values of the argument are given at equidistant intervals.

The formulae involve first and higher order differences which are calculated as shown below. Note that

$$\Delta y_i = y_{i+1} - y_i,\ \Delta^2 y_i = \Delta y_{i+1} - \Delta y_i \text{ etc....}$$

TABLE OF DIFFERENCES

x	y_x	Differences					
$\vdots$	$\vdots$						
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		$\mathbf{\Delta y_{-1}}$		$\mathbf{\Delta^3 y_{-2}}$		$\mathbf{\Delta^5 y_{-3}}$	
x_0	$\underline{\mathbf{y_0}}$		$\underline{\mathbf{\Delta^2 y_{-1}}}$		$\underline{\mathbf{\Delta^4 y_{-2}}}$		$\mathbf{\Delta^6 y_{-3}}$
		$\underline{\mathbf{\Delta y_0}}$		$\underline{\mathbf{\Delta^3 y_{-1}}}$		$\underline{\mathbf{\Delta^5 y_{-2}}}$	
x_1	$\underline{y_1}$		$\underline{\Delta^2 y_0}$		$\underline{\Delta^4 y_{-1}}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						
$\vdots$	$\vdots$						

Note : Differences underlined or in bold face have special significance only with respect to the explanation of certain formulae appearing in the next section. Those underlined appear in Bessel's formula and those in bold face in Stirling's formula,

b. Formulae

Let x be the value of the argument at which it is desired to interpolate and h the interval of the argument at which the ordinates are tabulated. Write $u=(x-x_0)/h$ where x_0 is a chosen value of the argument called the initial argument. Four main formulae are given depending on the nature of the subsequent arguments chosen in relation to initial argument x_0.

Newton's Forward Formula using arguments $x_0, x_1, x_2, \ldots$

$$y=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\ldots+\frac{u(u-1)\ldots(u-m+1)}{m!}\Delta^m y_0+\ldots$$

Note that the first $(m+1)$ terms give a polynomial of the m-th degree fitted to $y_0, y_1, \ldots, y_m$. The differences used are chosen from the principal diagonal (downwards) of the difference table starting from the initial ordinate y_0. The addition of the ordinate y_{m+1} brings in the correction term

$$\frac{u(u-1)\ldots(u-\overline{m+1}+1)}{(m+1)!}\Delta^{m+1}y_0$$

which involves the $(m+1)$th order difference at y_0.

Newton's Backward Formula using arguments $x_0, x_{-1}, x_{-2}, \ldots$

$$y=y_0+u\Delta y_{-1}+\frac{u(u+1)}{2!}\Delta^2 y_{-2}+\ldots+\frac{u(u+1)\ldots(u+m-1)}{m!}\Delta^m y_{-m}+\ldots$$

Note that the first $(m+1)$ terms give a polynomial of the m-th degree fitted to $y_0, y_{-1}, \ldots, y_{-m}$. The differences used are chosen from the principal diagonal (upwards) of the difference table starting from the initial ordinate y_0. The addition of the ordinate y_{-m-1} brings in the correction term

$$\frac{u(u+1)\ldots(u+m+1-1)}{(m+1)!}\Delta^{m+1}y_{-m-1}.$$

Stirling's Formula (for $-1/4<u<1/4$, using arguments) $x_0, x_{-1}, x_1, x_{-2}, x_2, \ldots$

$$y=y_0+u\frac{\Delta y_{-1}+\Delta y_0}{2}+\frac{u^2}{2!}\Delta^2 y_{-1}+\frac{u[u^2-1^2]}{3!}\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}+\frac{u^2[u^2-1^2]}{4!}\Delta^4 y_{-}+$$

$$\ldots+\frac{u[u^2-1^2]\ldots[u^2-(m-1)^2]}{(2m-1)!}\frac{\Delta^{2m-1}y_{-m}+\Delta^{2m-1}y_{-m+1}}{2}$$

$$+\frac{u^2[u^2-1^2)\ldots[u^2-(m-1)^2]}{2m!}\Delta^{2m}y_{-m}+\ldots$$

Note that the first $2m+1$ terms give the polynomial of degree $2m$ fitted to y_0, $y_{-1}, y_{+1}, y_{-2}, y_{+2}, \ldots, y_{-m}, y_{+m}$. The differences used are chosen as indicated (in bold face) in the table of differences. The addition of ordinates y_{-m-1} and y_{m+1} brings in the correction terms :

$$\frac{u[u^2-1^2]\ldots[u^2-m^2]}{(2m+1)!}\;\frac{\Delta^{2m+1}y_{-m-1}+\Delta^{2m+1}y_{-m}}{2}+\frac{u^2[u^2-1^2]\ldots[u^2-m^2]}{(2m+2)!}\Delta^{2m+2}y_{-m-1}.$$

Bessel's Formula (for $-1/4 < v < 1/4$, $v = u-\frac{1}{2}$) using arguments $x_0, x_1, x_{-1}, x_2 \ldots$

$$y=\frac{y_0+y_1}{2}+v\Delta y_0+\frac{\left[v^2-\left(\frac{1}{2}\right)^2\right]}{2!}\;\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\frac{v\left[v^2-\left(\frac{1}{2}\right)^2\right]}{3!}\Delta^3 y_{-1}+$$

$$+\frac{\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]}{4!}\;\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}+\frac{v\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]}{5!}\Delta^5 y_{-2}+\ldots$$

$$+\frac{\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]\cdots\left[v^2-\left(\frac{2m-3}{2}\right)^2\right]}{(2m-2)!}\;\frac{\Delta^{2m-2}y_{-m+1}+\Delta^{2m-2}y_{-m+2}}{2}$$

$$+\frac{v\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]\cdots\left[v^2-\left(\frac{2m-3}{2}\right)^2\right]}{(2m-1)!}\Delta^{2m-1}y_{-m+1}+\ldots$$

Note that the first $2m$ terms give the polynomial of degree $2m-1$ fitted to $y_0, y_1, y_{-1}, y_2 \ldots, y_{-m+1}, y_m$. The differences used are chosen as indicated (underlined) in the table of differences. The addition of ordinates y_{-m}, y_{m+1}, brings in the correction terms

$$\frac{\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]\cdots\left[v^2-\left(\frac{2m-1}{2}\right)^2\right]}{2m!}\;\frac{\Delta^{2m}y_{-m}+\Delta^{2m}y_{-m+1}}{2}$$

$$+\frac{v\left[v^2-\left(\frac{1}{2}\right)^2\right]\left[v^2-\left(\frac{3}{2}\right)^2\right]\cdots\left[v^2-\left(\frac{2m-1}{2}\right)^2\right]}{(2m+1)!}\Delta^{2m+1}y_{-m}.$$

c. Choice of formulae

Once the tabulated values to be used for interpolation are selected, it is immaterial which formula is used to obtain the desired value. For example if $f(7)$, $f(9)$, $f(11)$, $f(13)$, $f(15)$ are available and it is decided that all these values should be used for obtaining $f(11.4)$ one may stop with the fifth term of either Newton's Forward formula (with $x_0 = 7$ and $u = (11.4-7) \div 2 = 2.20$) or Stirling's formula (with $x_0 = 11$, $u = (11.4-11) \div 2 = 0.20$) the result obtained being the same, as the m-th degree polynomial whose values coincide with the values of the function at the $(m+1)$ selected arguments is unique. But in practice, after obtaining an interpolated

value based on a certain number of arguments, one may decide to consider a few more and compute the necessary correction to the value already obtained. The different formulae listed above are useful in different situations, depending on the positions of the additional arguments in relation to those already used. Newton's formulae requires the knowledge of additional tabulated values for arguments that are always on one side of x_0, moving further away from x_0 at each successive step. With Stirling's and Bessel's formulae the extra terms utilised will be chosen symmetrically from either side of x_0.

To begin with, the tabulated value of the argument close to x is chosen as x_0 giving the first approximation to y as y_0. If the subsequent values chosen are $x_1, x_2, \ldots$, Newton's Forward formula is used for step-by-step correction. If the subsequent values chosen are $x_{-1}, x_{-2}, \ldots$, Newton's Backward formula is used. If the subsequent values chosen are in pairs (x_{-1}, x_1), (x_{-2}, x_2), ... Stirling's formula is chosen. Or, one may begin with the pair (x_0, x_1) giving the first approximation to y as $(y_0+y_1)/2+(u-\frac{1}{2})\Delta y_0$, and then add the pair (x_{-1}, x_2) and so on. In such a case, Bessel's formula is used. *Note* that in each case, we add extra terms to the formula already obtained, as we bring in additional arguments either individually or in pairs.

d. Switching from one formula to another

It is not necessary to choose the arguments in only one particular manner throughout, in any given problem. If the tabular entries are limited on one side, it is not possible to carry out the central difference formula (Bessel or Stirling) to any sufficient length. Then the procedure is to use the central difference formula so long as the tabular entries permit, and then switch over to Newton's Forward or Backward formula, depending upon the direction in which subsequent values are chosen. The switching over is done only to obtain the correction terms by the new formulae without altering the approximation already obtained by the earlier formula. Thus, suppose in the numerical example considered above the fourth degree polynomial approximation obtained through Stirling's formula is found inadequate and further tabulated values are available only on one side of 11.4, say $f(17), f(19), \ldots$, then corrections to the interpolated value could be obtained from the sixth and succeeding terms of Newton's Forward formula.

$$\frac{u(u-1)\ldots(u-5)}{6!}\Delta^6 f(7)+\frac{u(u-1)\ldots(u-6)}{7!}\Delta^7 f(7)+\ldots$$

where $u=(x-7)/2=2.2$.

Some quadrature formulae

Numerical differentiation and integration are processes for approximate evaluation of derivatives and of definite integrals respectively when the function concerned is defined only by a table of ordinate values at discrete points. In either process, the function is first replaced by an interpolation polynomial which is conveniently differentiated or integrated. The numerical integration coefficients in Table 15.1 were obtained on the basis of Stirling's formula.

Simple quadrature formulae using the ordinates within the range of integration are given below.

(i) *Simpson's one third* rule (3 ordinates)

$$\int_a^b f(x)dx = \frac{b-a}{6}\left\{ f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right\}.$$

(ii) *Extension of Simpson's* rule by repeated application ($2n+1$ ordinates)

$$\int_a^b f(x)dx = \frac{h}{3}\{f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)+\ldots+f(b)\}$$

where $h=(b-a)/2n$ and n is an integer to be chosen.

(iii) *Three eighths* rule (4 ordinates)

$$\int_a^b f(x)dx = \frac{b-a}{16}\left\{2[f(a)+f(b)]+6\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]\right\}$$

(iv) *Hardy's* formula (5 ordinates)

$$\int_a^b f(x)dx = \frac{b-a}{6}\left\{0.28[f(a)+f(b)]+1.62\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]+2.2\,f\left(\frac{a+b}{2}\right)\right\}$$

(v) *Weddle's* rule (7 ordinates)

$$\int_a^b f(x)dx = 0.3\,h\,\{[f(a)+f(b)]+5[f(a+h)+f(a+5h)]$$

$$+[f(a+2h)+f(a+4h)]+6f(a+3h)]\},\quad h=(b-a)/6$$

(vi) *Shovelton's* formula (11 ordinates)

$$\int_a^b f(x)dx = \frac{5h}{126}\{8[f(a)+f(b)]+35[f(a+h)+f(a+3h)+f(a+7h)+f(a+9h)]+$$

$$+15[f(a+2h)+f(a+4h)+f(a+6h)+f(a+8h)]+36f(a+5h)\},\quad h=(b-a)/10.$$

For other formulae using external ordinates and the values of the multiplying co-efficients, see Table 15.1.

f. Summation formulae

Summation formulae given here are also useful for numerical integration

(i) *Euler-Maclaurin* sum formula.

$$f(a)+f(a+h)+\ldots+f(a+nh)$$

$$=\frac{1}{h}\int_a^{a+na} f(t)dt+\frac{1}{2}[f(a)+f(a+nh)]+\sum_{s=0} e_s\,h^{2s}\left[\frac{d^{2s+1}}{dt^{2s+1}}f(t)\right]_a^{a+nh}$$

where $e_s = B_{2(s+1)}/2(s+1)\,!$ and B_n are Bernoulli numbers given in Table 17.9. with the first few coefficients as follows, $e_0 = \frac{1}{12}$, $e_1 = -\frac{1}{720}$, $e_2 = \frac{1}{30240}$, $e_3 = -\frac{1}{1209600}$, $e_4 = \frac{1}{47900160}$. In practice, only the first two or three terms in the last summation need be considered

(ii) *Gregory's* sum formula

$$f(a)+f((a+h)+\ldots+f(a+nh)$$

$$=\frac{1}{h}\int_a^{a+nh} f(t)dt+\frac{1}{2}[f(a)+f(a+nh)]+\sum_{s=1} g_s\,[\Delta^s f(a+\overline{n-s}\,h)+(-1)^s\Delta^s f(a)]$$

where the coefficients g_s, are given by $\sum_{s=0}^{\infty} g_s\, t^s = t/\log(1-t)$ and the first few coefficients are as follows $g_1 = \frac{1}{12}$, $g_2 = \frac{1}{24}$, $g_3 = \frac{19}{720}$, $g_4 = \frac{3}{160}$, $g_5 = \frac{863}{60480}$, $g_6 = \frac{275}{24192}$

g. Solution of equations by algebraic methods.

(i) *Quadratic* equation

The roots of $ax^2+bx+c=0$ are

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a},\quad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

(ii) *Cubic* equation

The general cubic may be written

$$x^3+c_1x^2+c_2x+c_3 = 0. \qquad \ldots (1)$$

Setting $x = y-\frac{1}{3}c_1$, the equation takes the simple form

$$y^3+py+q = 0 \qquad \ldots (2)$$

where $p = c_2-\frac{1}{3}c_1^2$, $q = c_3-\frac{1}{3}c_1c_2+\frac{2}{27}c_1^3$. The roots of (2) are obtained by subtracting $c_1/3$ from each of the roots of (2). The roots of the reduced equation (2) are all real if $q^2+(4/27)p^3 < 0$. In such a case find a value of θ using Table 17.7 such that

$$\sin 3\theta = -4q/r^3 \qquad \ldots (3)$$

where $r = 2\sqrt{-p/3}$. If $\theta = \alpha$ is a solution of (3), then the three roots of (2) are

$$y_1 = -r\sin\alpha,\; y_2 = r\sin\left(\frac{\pi}{3}+\alpha\right),\; y_3 = r\sin\left(-\frac{\pi}{3}+\alpha\right). \qquad \ldots (4)$$

When $q^2+(4/27)p^3 > 0$, two of the roots are imaginary Let Q denote any one of the three values of

$$\{\tfrac{1}{2}(-q+\sqrt{q^2+(4/27)p^3})\}^{\frac{1}{3}} \qquad \ldots (5)$$

and ω be an imaginary cube root of unity. Then the three roots of (2) are

$$y_1 = Q-p/3Q,\; y_2 = \omega Q-\omega^2 p/3Q,\; y_3 = \omega^2 Q-\omega p/3Q. \qquad \ldots (6)$$

(iii) *Quartic* equation

The general quartic equation is written

$$ax^4+4bx^3+6cx^2+4dx+e = 0. \quad \ldots \quad (7)$$

First find a root of the cubic

$$s^3-3cs^2+(4bd-ae)s+3(ace-2ad^2-2eb^2) = 0 \quad \ldots \quad (8)$$

by the method indicated in (ii). Let s_1 be a root. Then compute $t_1 = (s_1-c)/2$,

$$m_1 = \sqrt{at_1+b^2-ac},\ n_1 = (2bt_1+bc-ad)/m_1. \quad \ldots \quad (9)$$

Then the four roots of the equation (7) are the roots of the two quadratics

$$\left.\begin{aligned} ax^2+2bx+c+2t_1 &= 2m_1x+n_1 \\ ax^2+2bx+c+2t_1 &= -(2m_1x+n_1) \end{aligned}\right\} \quad \ldots \quad (10)$$

Note: Polynomial equations of higher degree than 4 cannot be solved by algebriac reduction. The roots have to be found numerically by methods of successive approximations. The following book may be consulted for such methods.

J. B. Scarborough (1962). *Numerical mathematical analysis*. 5th. edition, Johns Hopkins Press, Baltimore.

PART II

TABLES WITH EXPLANATORY NOTES

1. THE BINOMIAL DISTRIBUTION

1.1. THE BINOMIAL COEFFICIENTS $\binom{n}{r}$

a. Introduction

Table 1.1 contains values of $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $n = 3(1)30$, $r = 2(1)\,[n/2]$.

The following formulae help to obtain $\binom{n}{r}$ for the values of r that are not given in Table 1.1.

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n \text{ and } \binom{n}{r} = \binom{n}{n-r}$$

b. Application

Table 1.1 can be used for computing individual terms $b(x \mid \pi, n) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$ of the binomial distribution.

Example. Let $n = 10$, $\pi = 0.73$; then $\theta = \pi/(1-\pi) = 2.70370$. The tabular scheme below shows the essential steps in computation. The first entry in column (3) is $(1-\pi)^{10} = 0.0^5205891$. The values which follow are obtained by successive multiplication with θ.

x	$\binom{n}{x}$*	$\pi^x(1-\pi)^{n-x}$	$b(x \mid \pi, n)$
(1)	(2)	(3)	(4)=(2)×(3)
0	1	$0.0^5205891$	0.0000
1	10	$0.0^5556667$	0.0001
2	45	$0.0^4150506$	0.0007
3	120	$0.0^4406923$	0.0049
4	210	$0.0^3110020$	0.0231
5	252	$0.0^3297461$	0.0751
6	210	$0.0^3804245$	0.1689
7	120	$0.0^2217444$	0.2609
8	45	$0.0^2587903$	0.2646
9	10	0.0158951	0.1589
10	1	0.0429756	0.0430

* From Table 1.1.

If accuracy upto k places of decimal is required in $b(x \mid \pi, n)$, it is advisable to calculate both $(1-\pi)$ and θ correct to $(k+2)$ significant digits and to retain $(k+2)$ significant digits at each stage in column (3)

The table of binomial coefficients is also useful in computing :

(i) multinomial coefficients, since

$$\frac{n!}{r_1!\, r_2! \cdots r_k!} = \binom{n}{r_1} \times \binom{n-r_1}{r_2} \times \cdots \binom{r_{k-1}+r_k}{r_{k-1}}$$

and (ii) the individual terms of the hypergeometric distribution, given by

$$\binom{a}{r} \times \binom{b}{n-r} \div \binom{a+b}{n}$$

TABLE 1.1. THE BINOMIAL COEFFICIENTS $\binom{n}{r}$

$[n = 3\,(1)\,30]$(1)

n	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$	$r = 10$	$r = 11$	$r = 12$	$r = 13$	$r = 14$	$r = 15$
3	3													
4	6													
5	10													
6	15	20												
7	21	35												
8	28	56	70											
9	36	84	126											
10	45	120	210	252										
11	55	165	330	462										
12	66	220	495	792	924									
13	78	286	715	1287	1716									
14	91	364	1001	2002	3003	3432								
15	105	455	1365	3003	5005	6435								
16	120	560	1820	4368	8008	11440	12870							
17	136	680	2380	6188	12376	19448	24310							
18	153	816	3060	8568	18564	31824	43758	48620						
19	171	969	3876	11628	27132	50388	75582	92378						
20	190	1140	4845	15504	38760	77520	125970	167960	184756					
21	210	1330	5985	20349	54264	116280	203490	293930	352716					
22	231	1540	7315	26334	74613	170544	319770	497420	646646	705432				
23	253	1771	8855	33649	100947	245157	490314	817190	1144066	1352078				
24	276	2024	10626	42504	134596	346104	735471	1307504	1961256	2496144	2704156			
25	300	2300	12650	53130	177100	480700	1081575	2042975	3268760	4457400	5200300			
26	325	2600	14950	65780	230230	657800	1562275	3124550	5311735	7726160	9657700	10400600		
27	351	2925	17550	80730	296010	888030	2220075	4686825	8436285	13037895	17383860	20058300		
28	378	3276	20475	98280	376740	1184040	3108105	6906900	13123110	21474180	30421755	37442160	40116600	
29	406	3654	23751	118755	475020	1560780	4292145	10015005	20030010	34597290	51895935	67863915	77558760	
30	435	4060	27405	142506	593775	2035800	5852925	14307150	30045015	54627300	86493225	119759850	145422675	155117520
n	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$	$r = 10$	$r = 11$	$r = 12$	$r = 13$	$r = 14$	$r = 15$

(1) For higher values of $n \leqslant 100$ see, *Tables of Binomial Coefficients* by J. C. P. Miller, Cambridge university Press, 1954.

Note: Values of $\binom{n}{r}$ are given for $r = 1, 2, \ldots, \left[\frac{n}{r}\right]$. For higher values of r observe that $\binom{n}{r} = \binom{n}{n-r}$

1.2. Individual Terms

a. Introduction

Table 1.2 gives, to five places of decimal, the values of $b(x|\ \pi,\ n) = \binom{n}{x}\pi^x(1-\pi)^{n-x}$, $x = 0(1)n$ for $n = 5(1)15$, and for the following selected values of π:

$$0.01,\ 0.02,\ 0.05,\ \frac{1}{16},\ 0.10,\ \frac{1}{9},\ \frac{3}{16},\ 0.20,\ \frac{1}{4},\ 0.30,\ \frac{1}{3},\ 0.40,\ \frac{7}{16},\ \frac{4}{9},\ \frac{1}{2}.$$

Note that since $b(x|\pi,\ n) = b(n-x|\,1-\pi,\ n)$ the coverage is automatically extended to the following additional values of π:

$$\frac{5}{9},\ \frac{9}{16},\ 0.60,\ \frac{2}{3},\ 0.70,\ \frac{3}{4},\ 0.80.\ \frac{13}{16},\ \frac{8}{9},\ 0.90,\ \frac{15}{16},\ 0.95,\ 0.98,\ 0.99.$$

The fractions correspond to values which occur in genetical studies. The values 0.01, 0.02, 0.05 correspond to critical levels generally used in tests of significance.

Table 1.2 has been obtained by differencing from a table of cumulative probabilities which is correct to 5 places of decimals. Some entries in this table are therefore in error by ± 1 in the last place ; this is indicated respectively by $-$ or $+$ sign against the entry.

b. Interpolation in Table 1.2

The following formula based on Taylor expansion could be used for interpolating at a specified value of π. Let π_0 be a tabular argument closest to π. Then

$$\begin{aligned} b(x|\pi,n) &= b(x|\pi_0,n) - dn\Delta\, b(x-1|\pi_0,n-1) \\ &\quad + \frac{d^2}{2!}\, n(n-1)\Delta^2\, b(x-2|\pi_0,n-2) + \cdots \\ &\quad + \frac{(-d)^k}{k!}\,(n)_k\,\Delta^k b\,(x-k|\pi_0,n-k) + R \end{aligned}$$

where

$$d = \pi - \pi_0,\ (n)_k = n(n-1)\ \ldots\ (n-k+1),$$

$$R = \frac{d^{k+1}}{(k+1)!}\,(n)_{k+1}\Delta^{k+1}\, b(x-k-1|\pi^*,\ n-k-1)$$

π^* being some intermediate value between π and π_0 and $\Delta,\ \Delta^2, \ldots$ represent differences of successive order taken with respect to x.

Example 1. To compute $b(2|\pi,\ n)$ for $n = 10$, $\pi = 0.27$. Here $\pi_0 = 0.25$, $d = 0.02$.

$$\begin{aligned} b(2|\,0.27,\ 10) &= 0.28156 - 0.02\times 10(0.30034-0.22526) \\ &\quad + \frac{(0.02)^2}{2}\times 10\times 9(0.31146-2\times 0.26697+0.10011) \\ &= 0.28156-0.2\times 0.07508+0.018(-0.12237) = 0.26434. \end{aligned}$$

Example 2. Out of 10 tests carried out on parallel sets of data, 3 were significant at 1% level. Are the results significant on the whole ?

To answer this question we have to determine the probability of obtaining 3 or more significant results, i.e. the probability of obtaining 3 or more successes in 10 trials when the probability of sucess at each trial is 0.01. Using Table 1.2, for $n = 10$ and $\pi = 0{\cdot}01$ the required probability is

$$1-[\Pr(x = 0)+\Pr(x = 1)+\Pr(x = 2)]$$

$$=1-(0.90438+0.09135+0.00416) = 0.00011$$

which is very small indicating that the results are significant on the whole. If only one were significant out of ten, then the probability

$$1-0.90438 = 0.09562$$

is not small enough to declare overall significance.

Table 1.2 is not exhaustive. For other values of π, for higher accuracy or for higher values of n, one may either consult more extensive tables or compute the values directly as illustrated in **1.1b**

c. Some other tables

1. National Bureau of Standards (1950): *Tables of the Binomial Probability Distribution*, Applied Math. Series No. 6, Washington.

   Individual terms and cumulative sums (cumulated from above) correct to 7 places for $\pi = 0.01$ (0.01) 0.50 and $n = 2(1)49$.

2. Romig, H. G. (1953) : 50–100 *Binomial Tables*, John Wiley & Sons, New York and London.

   Individual terms and cumulative sums (cumulated from below) correct to 6 places for $\pi = 0.01$ (0.01) 0.50 and $n = 50(1)100$.

3. U. S. Army Ordnance Corps (1952) : *Tables of the Cumulative Binomial Probabilities*, Ordnance Corps Pamphlet ORDP 20–1.

   Cumulative sums (cumulated from above) correct to 7 places for $\pi = 0.01(0.01)$ 0.50 and $n = 1(1)150$.

4. Harvard University, Computation Laboratory (1955) : *Tables of the Cumulative Binomial Probability Distribution*, The Annals of the Computation Laboratory of Harvard University, 35, Cambridge (Massachussetts).

   Cumulative sums (cumulated from above) correct to 5 places for $\pi = 0.01(0.01)$ $0.50\cdot$ $1/12(1/12)$ $5/12$; $1/16(1/16)$ $7/16$ and $n = 1(1)$ $50(2)$ $100(10)$ $200(20)$ $500(50)$ 1000.

5. Weintraub, S. (1963) : *Tables of the Cumulative Binomial Probability Distribution for Small Values of p*, The Free Press of Glencoe, Collier—Macmillan Ltd., London.

   Cumulative sums (cumulated from above) correct to 10 places for $\pi = 0.00001$, 0.0001 (0.0001) 0.0010 (0.0010) 0.1000 and $n = 1(1)100$.

TABLE 1.2. THE BINOMIAL DISTRIBUTION: INDIVIDUAL TERMS

[$n = 5(1)15$; selected values of π]

n	x	$\pi = 0.01$	$\pi = 0.02$	$\pi = 0.05$	$\pi = 1/16$	$\pi = 0.10$	$\pi = 1/9$	$\pi = 3/16$	$\pi = 0.20$
5	0	.95099	.90392	.77378	.72420	.59049	.55493	.35409	.32768
	1	.04803	.09224	.20363	.24140	.32805	.34683	.40857	.40960
	2	.00097	.00376	.02143	.03218+	.07290	.08671	.18857	.20480
	3	.00001	.00008	.00113	.00215	.00810	.01084	.04352	.05120
	4	—	—	.00003	.00007	.00045	.00067+	.00502	.00640
	5	—	—	—	—	.00001	.00002	.00023	.00032
6	0	.94148	.88584	.73509	.67893	.53144	.49327	.28770	.26214
	1	.05706	.10847	.23214−	.27158−	.35429	.36995	.39835	.39322
	2	.00144	.00554−	.03054	.04526	.09842	.11561	.22982	.24576
	3	.00002	.00015	.00214	.00402	.01458	.01927	.07072−	.08192
	4	—	—	.00009−	.00020	.00121+	.00181	.01224	.01536
	5	—	—	—	.00001	.00006−	.00009	.00113	.00154
	6	—	—	—	—	—	—	.00004	.00006
7	0	.93207	.86813	.69834	.63650	.47830	.43846	.23376	.20972
	1	.06590	.12401+	.25728	.29703	.37201	.38366−	.37760+	.36700
	2	.00200	.00760−	.04062	.05941	.12400	.14387	.26142	.27525
	3	.00003	.00025+	.00357−	.00660	.02296	.02997	.10055	.11469
	4	—	.00001	.00018+	.00044	.00255	.00375	.02320	.02867
	5	—	—	.00001	.00002	.00017	.00028	.00321	.00430
	6	—	—	—	—	.00001	.00001	.00025	.00036
	7	—	—	—	—	—	—	.00001	.00001
8	0	.92274	.85076	.66342	.59672	.43047	.38974	.18993	.16777
	1	.07457	.13890	.27934−	.31825	.38263+	.38975−	.35063	.33555−
	2	.00264	.00992	.05145+	.07426	.14881−	.17051	.28321−	.29360
	3	.00005	.00041	.00542	.00990	.03307	.04263	.13071	.14680
	4	—	.00001	.00035+	.00082+	.00459	.00666	.03770	.04587+
	5	—	—	.00002	.00005−	.00041	.00067	.00696	.00918
	6	—	—	—	—	.00002	.00004	.00081−	.00115
	7	—	—	—	—	—	—	.00005	.00008
9	0	.91352	.83375	.63025	.55942	.38742	.34644	.15432	.13422
	1	.08304+	.15314	.29854	.33566−	.38742	.38974	.32050	.30199
	2	.00336	.01250	.06285	.08951	.17219	.19488−	.29585	.30199
	3	.00008	.00059+	.00772	.01392	.04464	.05683+	.15930	.17616
	4	—	.00002	.00061	.00139	.00744	.01066	.05514	.06606
	5	—	—	.00003	.00010−	.00083	.00133	.01273	.01651
	6	—	—	—	—	.00006	.00011	.00195+	.00276−
	7	—	—	—	—	—	.00001	.00020−	.00029
	8	—	—	—	—	—	—	.00001	.00002
10	0	.90438	.81707	.59874	.52446	.34868	.30795	.12538	.10737
	1	.09135	.16675	.31512	.34964	.38742	.38493	.28934	.26844
	2	.00416−	.01532−	.07464−	.10489	.19371	.21652	.30047	.30199
	3	.00011	.00083	.01047+	.01865	.05739+	.07218−	.18491	.20133
	4	—	.00003	.00097−	.00218	.01117−	.01579	.07467	.08808
	5	—	—	.00006	.00017	.00148+	.00236+	.02068	.02642
	6	—	—	—	.00001	.00014	.00025	.00398	.00551
	7	—	—	—	—	.00001	.00002	.00052	.00078+
	8	—	—	—	—	—	—	.00005	.00008−
11	0	.89534	.80073	.56880	.49168	.31381	.27373	.10187	.08590
	1	.09948	.17976	.32931	.36057	.38355	.37638	.25860	.23622
	2	.00502	.01834	.08665+	.12019	.21308	.23524	.29839−	.29528
	3	.00016−	.00112	.01369−	.02403+	.07103	.08821	.20657	.22146
	4	—	.00005	.00144	.00321	.01578	.02205	.09534	.11073
	5	—	—	.00010+	.00030	.00245+	.00386	.03080	.03876
	6	—	—	.00001	.00002	.00028−	.00048	.00711	.00968+
	7	—	—	—	—	.00002	.00005−	.00117	.00173
	8	—	—	—	—	—	—	.00014	.00022
	9	—	—	—	—	—	—	.00001	.00002

Note: To obtain 5 decimal accuracy for individual terms add (subtract) 1 in the last place if there is +(−) sign against an entry. For obtaining cumulative probabilities to 5 decimal accuracy the entries have to be added ignoring the + and − signs.

TABLE 1.2 (*continued*). THE BINOMIAL DISTRIBUTION : INDIVIDUAL TERMS

[n = 5(1)15; selected values of π]

n	x	$\pi = 1/4$	$\pi = 0.30$	$\pi = 1/3$	$\pi = 0.40$	$\pi = 7/16$	$\pi = 4/9$	$\pi = 1/2$
5	0	.23730	.16807	.13169	.07776	.05631	.05292	.03125
	1	.39551	.36015	.32922	.25920	.21900	.21169	.15625
	2	.26367	.30870	.32921+	.34560	.34066	.33870	.31250
	3	.08789	.13230	.16461	.23040	.26496	.27096	.31250
	4	.01465	.02835	.04115	.07680	.10304	.10839−	.15625
	5	.00098	.00243	.00412	.01024	.01603	.01734	.03125
6	0	.17798	.11765	.08779	.04666	.03168	.02940	.01562
	1	.35596	.30252+	.26338−	.18662	.14782	.14113	.09375
	2	.29663	.32414	.32921+	.31104	.28743	.28225	.23438
	3	.13183+	.18522	.21948	.27648	.29808	.30107	.31250
	4	.03296	.05953	.08231−	.13824	.17388	.18064	.23437
	5	.00440−	.01021	.01646	.03686	.05410	.05780+	.09375
	6	.00024	.00073	.00137	.00410	.00701	.00771	.01563
7	0	.13348	.08235	.05853	.02799	.01782	.01633	.00781
	1	.31147−	.24707−	.20484+	.13064	.09701	.09147	.05469
	2	.31146	.31765	.30727	.26127	.22635	.21953	.16406
	3	.17303	.22689	.25606	.29031−	.29342	.29271	.27344
	4	.05768	.09724	.12803	.19353+	.22822	.23416	.27344
	5	.01154	.02501−	.03841	.07742−	.10650	.11240	.16406
	6	.00128	.00357	.00640	.01720	.02761	.02997	.05469
	7	.00006	.00022	.00046	.00164	.00307	.00343	.00781
8	0	.10011	.05765	.03902	.01680	.01002	.00907	.00391
	1	.26697	.19765	.15607	.08958	.06237−	.05808	.03125
	2	.31146	.29647+	.27313	.20901+	.16976+	.16261	.10937
	3	.20764	.25413−	.27313	.27870−	.26408	.26019−	.21875
	4	.08652	.13613+	.17071	.23224	.25674	.26018	.27344
	5	.02307	.04668	.06828	.12386	.15976−	.16652	.21875
	6	.00385	.01000	.01707	.04129	.06212+	.06660+	.10937
	7	.00036+	.00122	.00244	.00786	.01381	.01522	.03125
	8	.00002	.00007	.00015	.00066	.00134	.00152	.00391
9	0	.07508	.04035	.02601	.01008	.00564	.00504	.00195
	1	.22526−	.15565	.11706	.06046+	.03946	.03630	.01758
	2	.30034	.26683	.23411	.16125−	.12278	.11615	.07031
	3	.23359+	.26683	.27313	.25082	.22282	.21682	.16407−
	4	.11680	.17153	.20484+	.25082	.25995	.26018	.24609
	5	.03894−	.07352−	.10243−	.16722	.20218+	.20815	.24609
	6	.00865	.02100	.03414	.07432	.10484	.11101	.16407−
	7	.00123+	.00386	.00731+	.02123	.03495	.03806	.07031
	8	.00011−	.00041	.00092−	.00354	.00679+	.00761	.01758
	9	—	.00002	.00005	.00026	.00059	.00068	.00195
10	0	.05631	.02825	.01734	.00605	.00317	.00280	.00098
	1	.18772−	.12106	.08671	.04031	.02467−	.02241	.00976+
	2	.28156+	.23347	.19509	.12093	.08632+	.08066	.04395
	3	.25029	.26683	.26012	.21499	.17905	.17208	.11718+
	4	.14599+	.20012	.22761	.25082	.24371	.24091	.20508
	5	.05840	.10292	.13657−	.20066	.22746	.23127	.24610−
	6	.01622	.03676	.05690	.11148	.14743	.15418	.20507+
	7	.00309	.00900	.01626	.04247	.06552	.07049−	.11719
	8	.00039	.00145	.00304+	.01061+	.01911	.02114	.04395
	9	.00003	.00013+	.00034	.00158−	.00330	.00376	.00976+
	10	—	.00001	.00002	.00010	.00026	.00030	.00098
11	0	.04224	.01977	.01156	.00363	.00178	.00156	.00049
	1	.15486	.09322	.06359	.02660+	.01527−	.01369	.00537
	2	.25810	.19975	.15896	.08869−	.05935	.05477	.02685+
	3	.25810	.25682	.23845	.17736+	.13848	.13145	.08057
	4	.17207	.22014−	.23844+	.23649	.21542	.21032	.16113
	5	.08030	.13208	.16691	.22073−	.23457	.23555+	.22559
	6	.02677	.05660+	.08346	.14715	.18244	.18845−	.22559
	7	.00637	.01733	.02981	.07007	.10135+	.10768	.16113
	8	.00106	.00371	.00745	.02336	.03942	.04307	.08057
	9	.00012	.00053	.00124	.00519	.01022	.01149	.02685+
	10	.00001	.00005	.00012	.00069	.00159	.00184	.00537
	11	—	—	.00001	.00004	.00011	.00013	.00049

TABLE 1.2 (*continued*). THE BINOMIAL DISTRIBUTION: INDIVIDUAL TERMS

[$n = 5(1)15$; selected values of π]

n	x	$\pi = 0.01$	$\pi = 0.02$	$\pi = 0.05$	$\pi = 1/16$	$\pi = 0.10$	$\pi = 1/9$	$\pi = 3/16$	$\pi = 0.20$
12	0	.88638	.78472	.54036	.46095	.28243	.24332	.08277	.06872
	1	.10745−	.19217+	.34128	.36876	.37657	.36497	.22921	.20616
	2	.00596+	.02157	.09879	.13522−	.23013	.25092	.29093−	.28347
	3	.00021−	.00147	.01733	.03004+	.08523	.10455	.22379	.23622
	4	—	.00007	.00206−	.00451	.02131	.02940	.11619+	.13287+
	5	—	—	.00017	.00048	.00379	.00588	.04291−	.05315
	6	—	—	.00001	.00004	.00049	.00086	.01155	.01551−
	7	—	—	—	—	.00005	.00009	.00228	.00332
	8	—	—	—	—	—	.00001	.00033	.00052
	9	—	—	—	—	—	—	.00004−	.00006
13	0	.87752	.76902	.51334	.43214	.25419	.21628	.06725	.05498
	1	.11523	.20403	.35124−	.37453−	.36715+	.35146	.20176	.17867
	2	.00698	.02498	.11091+	.14980+	.24478−	.26359	.27935	.26800+
	3	.00026	.00187	.02141−	.03662	.09972	.12081	.23638	.24567
	4	.00001	.00010	.00281+	.00611−	.02770	.03775	.13637	.15355
	5	—	—	.00027	.00073	.00554	.00850−	05664+	.06909+
	6	—	—	.00002	.00007	.00082	.00142	.01743	.02304−
	7	—	—	—	—	.00009	.00017+	.00403−	.00575+
	8	—	—	—	—	.00001	.00002	.00069+	.00108
	9	—	—	—	—	—	—	.00009	.00015
	10	—	—	—	—	—	—	.00001	.00002
14	0	.86875	.75364	.48767+	.40513	.22877	.19225	.05464	.04398
	1	.12285	.21533	.35934	.37813−	.35586	.33644	.17654	.15393
	2	.00806+	.02856	.12294−	.16385	.25701	.27335	.26480	.25014
	3	.00033	.00233	.02588	.04370−	.11423	.13668	.24444−	.25014
	4	.00001	.00013	.00374+	.00801	.03490	.04698	.15512	17197
	5	—	.00001	.00040−	.00106+	.00776	.01175	.07159	.08599
	6	—	—	.00003	.00011	.00129	.00220	.02479−	.03224
	7	—	—	—	.00001	.00016	.00031	.00653+	.00921
	8	—	—	—	—	.00002	.00004−	.00132	.00202
	9	—	—	—	—	—	—	.00020	.00033+
	10	—	—	—	—	—	—	.00003−	.00005−
15	0	.86006	.73857	.46329	.37981	.20589	.17089	.04440	.03518
	1	.13031	.22609	.36576	.37981	.34315	.32041+	.15368	.13195−
	2	.00921	.03230	.13475	.17725	.26690	.28037−	.24825	.23089+
	3	.00041−	.00286	.03073	.05120	.12850+	.15186	.24825	.25014
	4	.00001	.00017	.00486−	.01025−	.04284	.05695	.17187	.18761−
	5	—	.00001	.00056	.00150	.01047	.01566	.08726	.10318
	6	—	—	.00005	.00016+	.00194	.00326	.03356	.04299
	7	—	—	—	.00002−	.00028	.00053−	.00996	.01382
	8	—	—	—	—	.00003	.00006+	.00229+	.00346−
	9	—	—	—	—	—	.00001	.00042−	.00067
	10	—	—	—	—	—	—	.00005+	.00010
	11	—	—	—	—	—	—	.00001	.00001

TABLE 1.2 (*continued*). THE BINOMIAL DISTRIBUTION : INDIVIDUAL TERMS

[$n = 5(1)15$: selected values of π]

n	x	$\pi = 1/4$	$\pi = 0.30$	$\pi = 1/3$	$\pi = 0.40$	$\pi = 7/16$	$\pi = 4/9$	$\pi = 1/2$
	0	.03168	.01384	.00771	.00218	.00100	.00086	.00024
	1	.12670+	.07119−	.04624	.01741	.00937−	.00830	.00293
	2	.23230−	.16779	.12717	.06385	.04006	.03652−	.01612−
	3	.25810	.23970	.21195	.14190−	.10386	.09737	.05371
	4	.19358	.23114	.23845	.21284	.18176	.17526	.12085
	5	.10324	.15849+	.19076	.22703	.22619	.22434	.19336
	6	.04015	.07925	.11127	.17658	.20525	.20938	.22558+
	7	.01147	.02911	.04769	.10090	.13683	.14358	.19336
	8	.00239	.00780	.01490	.04204	.06651+	.07179	.12085
	9	.00035	.00148+	.00332−	.01246	.02300−	.02552	.05371
	10	.00004	.00019	.00049+	.00249	.00536+	.00613	.01612−
	11	—	.00002−	.00005	.00030	.00076	.00089	.00293
	12	—	—	—	.00002	.00005	.00006	.00024
13	0	.02376	.00969	.00514	.00131	.00056	.00048	.00012
	1	.10295	.05398	.03340	.01132	.00571	.00499	.00159
	2	.20589+	.13881	.10019+	.04527+	.02663	.02398−	.00952
	3	.25165	.21813	.18369	.11068	.07595	.07032	.03491
	4	.20971	.23370+	.22962−	.18446	.14768	.14065−	.08728
	5	.12583	.18029	.20665	.22136−	.20675	.20252+	.15711−
	6	.05592	.10302	.13777	.19676	.21441	.21603	.20947
	7	.01864	.04416−	.06889−	.13117	.16677−	.17283−	.20947
	8	.00466	.01419	.02583	.06559	.09727+	.10369	.15711−
	9	.00086	.00338	.00717+	.02429	.04204−	.04609	.08728
	10	.00012	.00058	.00144	.00647+	.01307+	.01475	.03491
	11	.00001	.00007	.00019+	.00118	.00278−	.00321+	.00952
	12	—	—	.00002	.00013	.00036	.00043	.00159
	13	—	—	—	.00001	.00002	.00003	.00012
14	0	.01782	.00678	.00343	.00078	.00032	.00027	.00006
	1	.08315	.04070−	.02397+	.00732−	.00345+	.00298+	.00086−
	2	.18016	.11336	.07793	.03169	.01748	.01554	.00555
	3	.24021	.19433	.15586	.08452	.05437	.04973−	.02222
	4	.22019	.22903	.21431	.15495	.11630	.10939	.06109+
	5	.14680	.19632−	.21431	.20659+	.18091	.17502	.12220−
	6	.07340	.12620	.16073+	.20660	.21106	.21003	.18328+
	7	.02796	.06181	.09184+	.15741	.18761	.19203	.20948−
	8	.00816	.02318	.04019−	.09182	.12767+	.13442	.18328+
	9	.00181	.00662	.01339	.04081	.06621−	.07169	.12220−
	10	.00030	.00142	.00335	.01360	.02574+	.02867+	.06109+
	11	.00004	.00022	.00061	.00330	.00729−	.00834	.02222
	12	—	.00003−	.00007+	.00055	.00141+	.00167	.00555
	13	—	—	.00001	.00006	.00017	.00021	.00086−
	14	—	—	—	—	.00001	.00001	.00006
15	0	.01336	.00475	.00228	.00047	.00018	.00015	.00003
	1	.06682	.03052	.01713	.00470	.00208	.00178	.00046
	2	.15591	.09156	.05995	.02194	.01135−	.00996	.00320
	3	.22520	.17004	.12988	.06339	.03823	.03453	.01389
	4	.22520	.21862	.19482	.12678	.08920+	.08287	.04165+
	5	.16514+	.20613	.21431−	.18594	.15264	.14585	.09165−
	6	.09175	.14724	.17859	.20659+	.19787	.19447	.15274
	7	.03932	.08113	.11481	.17709−	.19787	.20003	.19638
	8	.01311	.03477	.05740	.11805+	.15390	.16002	.19638
	9	.00340	.01159	.02233−	.06122−	.09309+	.09957	.15274
	10	.00067+	.00298	.00669+	.02448+	.04345	.04780−	.09165−
	11	.00011−	.00058	.00152	.00742	.01536	.01738	.04165+
	12	.00001	.00008	.00026−	.00165	.00398	.00463	.01389
	13	—	.00001	.00003	.00025	.00072−	.00086	.00320
	14	—	—	—	.00003−	.00008	.00009+	.00046
	15	—	—	—	—	—	.00001	.00003

1.3. Confidence Intervals For The Binomial Proportion

a. Introduction

Table 1.3 furnishes *two sided* $95^{+}\%$ and $99^{+}\%$ confidence limits for the unknown binomial proportion π, corresponding to the number of trials n and the observed value of x.

These confidence limits have the property, that compared to any other system of limits with confidence coefficients not less than 95%, and 99%, the total length of confidence intervals corresponding to $x = 0, 1, \ldots, n$ is the least. For details see Crow (1956, Biometrika, 43, 423-435) and Sterne (1954, Biometrika, 41, 275-278).

The confidence limits given in Table 1.3 are correct to three places of decimal and are for $n = 1(1)30$ and $x = 0(1)[n/2]$. If x is greater than $[n/2]$, $n-x$ would be $\leqslant [n/2]$, and the table can be read for confidence limits for the complementary proportion $(1-\pi)$ from which the confidence limits for π are obtained.

Example. Suppose $n = 25$ and $x = 14$. Then $n-x = 11$. Entering Table 1.3 with $x = 11$ and $n = 25$ the 95% limits for $1-\pi$ are seen to be (0.238, 0.664) which means that the 95% limits for π would be $(1-0.664,\ 1-0.238) = (0.336,\ 0.762)$.

b. One sided confidence intervals

The $100\alpha\%$ lower bound for π is the smallest value of π, satisfying the inequality $P(d;\ \pi, n) = \sum_{x=d}^{n} b(x \mid \pi, n) \geqslant 1-\alpha$, where d is the observed value of x.

Since
$$Q(d; \pi, n) = \frac{1}{B(d, n-d+1)} \int_0^{\pi} t^{d-1} (1-t)^{n-d}\, dt,$$

this lower bound is seen to be the lower $100(1-\alpha)$ % point of the beta distribution with parameters d and $n-d+1$ respectively. (See Table 6.2 for percentage points of the beta distribution). Similarly the $100\alpha\%$ upper bound on π is given by the upper $100(1-\alpha)$ % point of the beta distribution with parameters $d+1$ and $n-d$ respectively.

c. Tests of significance

Table 1.3 can also be used for testing a simple hypothesis on π, when alternatives are both-sided. If $x = d$ be the observed value of x in n trials, we find, from Table 1.3 the corresponding $100\alpha\%$ confidence interval for π. A null hypothesis which assigns a value of π outside the confidence interval is rejected at the $100(1-\alpha)\%$ level of significance.

Example. 18 tosses of a coin result in 5 heads. Is this compatible with the hypothesis that the coin is unbiased ?

Here $n = 18$, $x = 5$. The corresponding 95% confidence interval for π being (0.116, 0.556), the hypothesis $\pi = 0.5$ cannot be rejected at the 5% level of significance.

Table 6.2 (percentage points of the beta distribution) can be similarly used for testing a simple hypothesis on π, when alternatives are one-sided. Suppose in the above example the hypothesis $\pi = 0.5$ is to be tested against alternatives $\pi < 0.5$. The 95% upper bound for π (which is same as the upper 5% point of $B(6, 13)$) $= 0.4978$. Since the hypothetical value exceeds this value, the hypothesis stands rejected at the 5% level of significance.

TABLE 1.3. CONFIDENCE INTERVALS(1) FOR THE BINOMIAL PROPORTION

Confidence coefficient : 95+%

n	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$	$x=12$	$x=13$	$x=14$	$x=15$
1	.950															
	.000															
2	.776	.975														
	.000	.025														
3	.632	.865														
	.000	.017														
4	.527	.751	.902													
	.000	.013	.098													
5	.500	.657	.811													
	.000	.010	.076													
6	.402	.598	.729	.847												
	.000	.009	.063	.153												
7	.377	.554	.659	.775												
	.000	.007	.053	.129												
8	.315	.500	.685	.711	.807											
	.000	.006	.046	.111	.193											
9	.289	.443	.558	.711	.749											
	.000	.006	.041	.098	.169											
10	.267	.397	.603	.619	.733	.778										
	.000	.005	.037	.087	.150	.222										
11	.250	.369	.500	.631	.667	.750										
	.000	.005	.033	.079	.135	.200										
12	.236	.346	.450	.550	.654	.706	.764									
	.000	.004	.030	.072	.123	.181	.236									
13	.225	.327	.434	.520	.587	.673	.740									
	.000	.004	.028	.066	.113	.166	.224									
14	.206	.312	.389	.500	.611	.629	.688	.794								
	.000	.004	.026	.061	.104	.153	.206	.206								
15	.191	.302	.369	.448	.552	.631	.668	.706								
	.000	.003	.024	.057	.097	.142	.191	.191								
16	.178	.272	.352	.429	.500	.571	.648	.728	.728							
	.000	.003	.023	.053	.090	.132	.178	.178	.272							
17	.166	.254	.337	.417	.489	.544	.594	.663	.746							
	.000	.003	.021	.050	.085	.124	.166	.166	.253							
18	.157	.242	.325	.381	.444	.556	.619	.625	.675	.758						
	.000	.003	.020	.047	.080	.116	.156	.157	.236	.242						
19	.150	.232	.316	.365	.426	.500	.574	.635	.655	.688						
	.000	.003	.019	.044	.075	.110	.147	.150	.222	.232						
20	.143	.222	.293	.351	.411	.467	.533	.589	.649	.707	.707					
	.000	.003	.018	042	.071	.104	.140	.143	.209	.222	.293					
21	.137	.213	.276	.338	.398	.455	.506	.551	.602	.662	.724					
	.000	.002	.017	.040	.068	.099	.132	.137	.197	.213	.276					
22	.132	.205	.264	.326	.389	.424	.500	.576	.582	.617	.674	.736				
	.000	.002	.016	.038	.065	.094	.126	.132	.187	.205	.260	.264				
23	.127	.198	.255	.317	.360	.409	.457	.543	.591	.640	.640	.683				
	.000	.002	.016	.037	.062	.090	.120	.127	.178	.198	.247	.255				
24	.122	.191	.246	.308	.347	.396	.443	.500	.557	.604	.653	.661	.692			
	.000	.002	.015	.035	.059	.086	.115	.122	.169	.191	.234	.246	.308			
25	.118	.185	.238	.303	.336	.384	.431	.475	.525	.569	.616	.664	.683			
	.000	.002	.014	.034	.057	.082	.110	.118	.161	.185	.222	.238	.296			
26	.114	.180	.230	.282	.325	.374	.421	.465	.506	.542	.579	.626	.675	.718		
	.000	.002	.014	.032	.054	.079	.106	.114	.154	.180	.212	.230	.282	.282		
27	.110	.175	.223	.269	.316	.364	.415	.437	.500	.563	.570	.598	.636	.684		
	.000	.002	.013	.031	.052	.076	.101	.110	.148	.175	.202	.223	.269	.269		
28	.106	.170	.217	.259	.307	.357	.384	.424	.463	.537	.576	.616	.619	.645	.693	
	.000	.002	.013	.030	.050	.073	.098	.106	.142	.170	.192	.217	.258	.259	.307	
29	.103	.166	.211	.251	.299	.339	.374	.413	.451	.500	.549	.587	.626	.661	.661	
	.000	.002	.012	.029	.049	.070	.094	.103	.136	.166	.184	.211	.247	.251	.299	
30	.100	.163	.205	.244	.292	.324	.364	.403	.440	.476	.524	.560	.597	.636	.676	.676
	.000	.002	.012	.028	.047	.068	.091	.100	.131	.163	.175	.205	.236	.244	.292	.324
	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$	$x=12$	$x=13$	$x=14$	$x=15$

(1) For a different type of confidence intervals see Table 11 in *Statistical Tables and Formulae* by A. Hald, John Wiley and Sons, New York, 1952.

TABLE 1.3. CONFIDENCE INTERVALS FOR THE BINOMIAL PROPORTION

Confidence coefficient : 99+%.

n	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$	$x=12$	$x=13$	$x=14$	$x=15$
1	.990															
	.000															
2	.900	.995														
	.000	.005														
3	.785	.941														
	.000	.003														
4	.684	.859	.958													
	.000	.003	.042													
5	.602	.778	.894													
	.000	.002	.033													
6	.536	.706	.827	.915												
	.000	.002	.027	.085												
7	.500	.643	.764	.858												
	.000	.001	.023	.071												
8	.451	.590	.707	.802	.879											
	.000	.001	.020	.061	.121											
9	.402	.598	.656	.750	.829											
	.000	.001	.017	.053	.105											
10	.376	.512	.624	.703	.782	.850										
	.000	.001	.016	.048	.093	.150										
11	.359	.500	.593	.660	.738	.806										
	.000	.001	.014	.043	.084	.134										
12	.321	.445	.555	.679	.698	.765	.825									
	.000	.001	.013	.039	.076	.121	.175									
13	.302	.429	.523	.594	.698	.727	.787									
	.000	.001	.012	.036	.069	.111	.159									
14	.286	.392	.500	.608	.636	.714	.751	.805								
	.000	.001	.011	.033	.064	.102	.146	.195								
15	.273	.373	.461	.539	.627	.672	.727	.771								
	.000	.001	.010	.031	.059	.094	.135	.179								
16	.264	.357	.451	.525	.579	.643	.705	.739	.788							
	.000	.001	.010	.029	.055	.088	.125	.166	.212							
17	.242	.346	.413	.500	.587	.620	.662	.758	.758							
	.000	.001	.009	.027	.052	.082	.117	.155	.197							
18	.228	.318	.397	.466	.534	.603	.682	.686	.772	.774						
	.000	.001	.008	.025	.049	.077	.110	.145	.184	.226						
19	.218	.305	.383	.455	.515	.564	.617	.695	.707	.782						
	.000	.001	.008	.024	.046	.073	.103	.137	.173	.212						
20	.209	.293	.375	.424	.500	.576	.601	.637	.707	.726	.791					
	.000	.001	.008	.023	.044	.069	.098	.129	.163	.200	.209					
21	.201	.283	.347	.409	.466	.534	.591	.653	.661	.717	.743					
	.000	.000	.007	.022	.041	.065	.092	.122	.155	.189	.201					
22	.194	.273	.334	.396	.454	.505	.550	.604	.666	.682	.727	.758				
	.000	.000	.007	.021	.039	.062	.088	.116	.147	.179	.194	.242				
23	.187	.265	.323	.386	.429	.500	.571	.580	.616	.677	.702	.735				
	.000	.000	.007	.020	.038	.059	.084	.111	.140	.171	.187	.229				
24	.181	.259	.313	.364	.416	.464	.536	.584	.636	.638	.687	.720	.743			
	.000	.000	.006	.019	.036	.057	.080	.106	.133	.163	.181	.216	.257			
25	.175	.245	.305	.352	.403	.451	.500	.549	.597	.648	.658	.695	.755			
	.000	.000	.006	.018	.034	.054	.077	.101	.127	.155	.175	.205	.245			
26	.170	.234	.298	.342	.393	.442	.487	.526	.562	.607	.658	.678	.702	.766		
	.000	.000	.006	.017	.033	.052	.073	.097	.122	.149	.170	.195	.234	.234		
27	.166	.225	.297	.332	.384	.419	.461	.539	.581	.587	.617	.668	.702	.716		
	.000	.000	.006	.017	.032	.050	.070	.093	.117	.143	.166	.185	.224	.225		
28	.162	.218	.272	.323	.364	.408	.449	.500	.551	.592	.636	.636	.677	.728	.728	
	.000	.000	.005	.016	.031	.048	.068	.089	.112	.137	.162	.175	.214	.218	.272	
29	.160	.211	.263	.316	.354	.397	.438	.477	.523	.562	.603	.646	.654	.684	.737	
	.000	.000	.005	.015	.030	.046	.065	.086	.108	.132	.157	.165	.206	.211	.260	
30	.151	.206	.256	.310	.345	.388	.430	.469	.505	.538	.570	.612	.655	.671	.692	.744
	.000	.000	.005	.015	.028	.045	.063	.083	.104	.127	.151	.151	.198	.206	.249	.256
n	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$	$x=8$	$x=9$	$x=10$	$x=11$	$x=12$	$x=13$	$x=14$	$x=15$

2.1 Cumulative Probability

a. Introduction

Let $p(x\,|\,\lambda) = e^{-\lambda}\dfrac{\lambda^x}{x!}$ (probability for observation x) and $P(x\,|\,\lambda) = \sum\limits_{r=0}^{x} p(r\,|\,\lambda)$ (cumulative probability for observations up to x). Table 2.1 gives the values $P(x\,|\,\lambda)$, $x = 0, 1, 2, \ldots$ for $\lambda = 0.02(0.02)\,0.10;\ 0.15(0.05)\ 1.00;\ 1.1(0.1)2.0,\ 2.2(0.2)7.8;\ 8.0(0.5)\ 15.0;\ 16(1)25$. Table 2.1a gives the values of $P(x\,|\,\lambda)$ for small values of $\lambda = 0.0005$; $0.001(0.001)0.009$. The individual terms $p(x\,|\,\lambda)$ can be easily obtained from these tables by the relation $p(x\,|\,\lambda) = P(x\,|\,\lambda) - P(x-1\,|\,\lambda)$

b. Interpolation in Table 2.1

For purposes of interpolation (λ-wise) between the tabulated values the following formula based on Taylor expansion will be found useful. Let the values of $P(x\,|\,\lambda)$ be required for a given λ, and λ_0 stand for the tabular argument closest to λ, and let $d = \lambda - \lambda_0$. Then

$$P(x\,|\,\lambda) = P(x\,|\,\lambda_0) - d\Delta P(x-1\,|\,\lambda_0) + \frac{d^2}{2!}\,\Delta^2 P(x-2\,|\,\lambda_0) + \ldots + (-1)^k\,\frac{d^k}{k!}\,\Delta^k P(x-k\,|\,\lambda_0) + R$$

where $\Delta, \Delta^2 \ldots$ are the 1st, 2nd...order differences taken with respect x and $R = (-1)^{k+1}\dfrac{d^{k+1}}{(k+1)!}\,P(x-k-1\,|\,\lambda^*)$ where λ^* is some value lying between λ and λ_0. It will be thus possible to judge, by inspection of the tabulated values, the maximum possible magnitude for the error R.

Example

To compute $P(4\,|\,5.1)$

From Table 2.1, $\lambda_0 = 5.0$ so that $d = 0.1$. Omitting terms involving second and higher order differences

$$P(4\,|\,5.1) = 0.440 - 0.1(0.440 - 0.265) = 0.4225$$

$$R = (-1)^2\,\frac{(0.1)^2}{2!}\,\Delta^2 P(2\,|\,\lambda^*) \text{ where } 5.0 < \lambda^* < 5.1$$

Since $\Delta^2 P(2\,|\,5) = 0.035$, $\Delta^2 P(2\,|\,5.2) = 0.039$, $\Delta^2 P(2\,|\,5.4) = 0.042$ and Δ^2 increases with λ

$$0 < R < \frac{(0.1)^2}{2} \times 0.039 = 0.000195.$$

When the interpolation for a given value of λ has to be repeated, as for instance in fitting a Poisson distribution, the above formula may be written as follows

(i) linear interpolation

$$P(x\,|\,\lambda) = (1-d)P(x\,|\,\lambda_0) + dP(x-1\,|\,\lambda_0)$$

(ii) quadratic interpolation

$$P(x\,|\,\lambda) = (1-d+d^2/2)\,P(x\,|\,\lambda_0) + (d-d^2)P(x-1\,|\,\lambda_0) + \frac{d^2}{2}\,P(x-2\,|\,\lambda_0)$$

Example

Fit a Poisson law to the frequency distribution of number of defects in 377 metal sheets produced in a factory.

No. of defects	Number of sheets	Poisson frequency
0	181	190.4
1	142	129.9
2	47	44.6
2	6	10.0
4	1	1.8
5 and above	—	0.3
Total	377	377.0

The mean of the observed frequency distribution is 0.684 which provides an estimate for λ of the Poisson distribution to be fitted. The nearest tabular argument λ_0 in Table 2.1 is 0.70 so that $d = -0.016$. Using the formula for linear interpolation $P(0\,|\,0.684) = 0.504952$, $P(1\,|\,0.6849) = 0.849552$, $P(2\,|\,0.684) = 0.967952$, $P(3\,|\,0.684) = 0.994448$, $P(4\,|\,0.684) = 0.999080$ and $P(5\,|\,0.684) = 1.0$. The values of $p(x\,|\,0.684)$ for $x = 0, 1, 2, 3, 4, 5$ are 0.5050, 0.3446, 0.1184, 0.0265, 0.0046, 0.0009. The Poisson frequencies obtained by multiplying these by 377 are shown in the last column above.

Table 2.1 is particularly useful in working out the operating charateristic curves of acceptance sampling plans by attributes. Poisson distribution is also applicable in a variety of industrial and other situations such as number of accidents, defects in cloth, defects in castings, frequency of breakdown of machines, demand for spares etc.

c. Some other tables of the Poisson distribution

1. Molina, E. C. (1942): *Poisson's Exponential Binomial Limit*, Van Nostrand Book Company, New York.
   Individual terms and cumulative terms of the distribution, correct to 6 and 7 places for $\lambda = 0.001\ (0.001)\ 0.01\ (0.01)\ 0.3\ (0.1)\ 15\ (1)\ 100$.
2. Kitagawa, Tosio (1952): *Tables of Poisson Distribution*, Baifukan, Tokyo.
   Individual terms, correct to 7 and 8 decimal places for $\lambda = 0.001\ (0.001)\ 1\ (0.01)\ 10.00$.
3. Pearson, E. S. and Hartley, H. O. (Eds.) (1957): *Biometrika Tables for Statisticians*. Biometrika Trust, Cambridge University Press.
   Table 7: Probability integral of the χ^2 distribution and the cumulative sum of the Poisson distribution correct to five decimal places for $\lambda = 0.0005 (0.0005)\ 0.005, 0.005 (0.005)\ 0.05, 0.05 (0.05)\ 1.0, 1.1 (0.1)\ 5.0, 5.25 (0.25)\ 10.0, 10.5 (0.5)\ 20.0, 21 (1.0)\ 60.$ and Table 39: Individual terms of the Poisson distribution, $\lambda = 0.1 (0.1)\ 15.0$.

TABLE 2.1. THE POISSON DISTRIBUTION-CUMULATIVE PROBABILITIES

Entries in body of table give the probability of x or less when the expected number is that given in the left margin of the table

$\lambda \diagdown x$	0	1	2	3	4	5	6	7	8
0.02	.980	1.000							
0.04	.961	.999	1.000						
0.06	.942	.998	1.000						
0.08	.923	.997	1.000						
0.10	.905	.995	1.000						
0.15	.861	.990	999	1.000					
0.20	.819	.982	.999	1.000					
0.25	.779	.974	.998	1.000					
0.30	.741	.963	.996	1.000					
0.35	.705	.951	.994	1.000					
0.40	.670	.938	.992	.999	1.000				
0.45	.638	.925	.989	.999	1.000				
0.50	.607	.910	.986	.998	1.000				
0.55	.577	.894	.982	.998	1.000				
0.60	.549	.878	.977	.997	1.000				
0.65	.522	.861	.972	.996	.999	1.000			
0.70	.497	.844	.966	.994	.999	1.000			
0.75	.472	.827	.959	.993	.999	1.000			
0.80	.449	.809	.953	.991	999	1.000			
0.85	.427	.791	.945	.989	.998	1.000			
0.90	.407	.772	.937	.987	.998	1.000			
0.95	.387	.754	.929	.984	.997	1.000			
1.00	.368	.736	.920	.981	.996	.999	1.000		
1.1	.333	.699	.900	.974	.995	.999	1.000		
1.2	.301	.663	.879	.966	.992	.998	1.000		
1.3	.273	.627	.857	.957	.989	.998	1.000		
1.4	.247	.592	.833	.946	.986	.997	.999	1.000	
1.5	.223	.558	.809	.934	.981	.996	.999	1.000	
1.6	.202	.525	.783	.921	.976	.994	.999	1.000	
1.7	.183	.493	.757	.907	.970	.992	.998	1.000	
1.8	.165	.463	.731	.891	.964	.990	.997	.999	1.000
1.9	.150	.434	.704	.875	.956	.987	.997	.999	1.000
2.0	.135	.406	.677	.857	.947	.983	.995	.999	1.000

λ \ x	0	1	2	3	4	5	6	7	8	9
2.2	.111	.355	.623	.819	.928	.975	.993	.998	1.000	
2.4	.091	.308	.570	.779	.904	.964	.988	.997	.999	1.000
2.6	.074	.267	.518	.736	.877	.951	.983	.995	.999	1.000
2.8	.061	.231	.409	.692	.848	.935	.976	.992	.998	.999
3.0	.050	.199	.423	.647	.815	.916	.966	.988	.996	.999
3.2	.041	.171	.380	.603	.781	.895	.955	.983	.994	.998
3.4	.033	.147	.340	.558	.744	.871	.942	.977	.992	.997
3.6	.027	.126	.303	.515	.706	844	.927	.969	.988	.996
3.8	.022	.107	.269	.473	.668	.816	.909	.960	.984	.994
4.0	.018	.092	.238	.433	.629	.785	.889	.949	.979	.992
4.2	.015	.078	.210	.395	.590	.753	.867	.936	.972	.989
4.4	.012	.066	.185	.359	.551	.720	.844	.921	.964	.985
4.6	.010	.056	.163	.326	.513	.686	.818	.905	.955	.980
4.8	.008	.048	.143	.294	.476	.651	.791	.887	.944	.975
5.0	.007	.040	.125	.265	.440	.616	.762	.867	.932	.968
5.2	.006	.034	.109	.238	.406	.581	.732	.845	.918	.960
5.4	.005	.029	.095	.213	.373	.546	.702	.822	.903	.951
5.6	.004	.024	.082	.191	.342	.512	.670	.797	.886	.941
5.8	.003	.021	.072	.170	.313	.478	.638	.771	.867	.929
6.0	.002	.017	.062	.151	.285	.446	.606	.744	.847	.916

λ \ x	10	11	12	13	14	15	16
2.8	1.000						
3.0	1.000						
3.2	1.000						
3.4	.999	1.000					
3.6	.999	1.000					
3.8	.998	.999	1.000				
4.0	.997	.999	1.000				
4.2	.996	.999	1.000				
4.4	.994	.998	.999	1.000			
4.6	.992	.997	.999	1.000			
4.8	.990	.996	.999	1.000			
5.0	.986	.995	.998	.999	1.000		
5.2	.982	.993	.997	.999	1.000		
5.4	.977	.990	.996	.999	1.000		
5.6	.972	.988	.995	.998	.999	1.000	
5.8	.965	.984	.993	.997	.999	1.000	
6.0	.957	.980	.991	.996	.999	.999	1.000

λ \ x	0	1	2	3	4	5	6	7	8	9
6.2	.002	.015	.054	.134	.259	.414	.574	.716	.826	.902
6.4	.002	.012	.046	.119	.235	.384	.542	.687	.803	.886
6.6	.001	.010	.040	.105	.213	.355	.511	.658	.780	.869
6.8	.001	.009	.034	.093	.192	.327	.480	.628	.755	.850
7.0	.001	.007	.030	.082	.173	.301	.450	.599	.729	.830
7.2	.001	.006	.025	.072	.156	.276	.420	.569	.703	.810
7.4	.001	.005	.022	.063	.140	.253	.392	.539	.676	.788
7.6	.001	.004	.019	055	.125	.231	.365	.510	.648	.765
7.8	.000	.004	.016	.048	.112	.210	.338	.481	.620	.741
8.0	.000	.003	.014	.042	.100	.191	313	.453	.593	.717
8.5	.000	.002	.009	.030	.074	.150	.256	.386	.523	.653
9.0	.000	.001	.006	.021	.055	.116	.207	.324	.456	.587
9.5	.000	.001	.004	.015	.040	.089	.165	.269	.392	.522
10.0	.000	.000	.003	.010	.029	067	.130	.220	.333	.458

λ \ x	10	11	12	13	14	15	16	17	18	19
6.2	.949	.975	.989	.995	.998	.999	1.000			
6.4	.939	.969	.986	.994	.997	.999	1.000			
6.6	.927	.963	.982	.992	.997	.999	.999	1.000		
6.8	.915	.955	.978	.990	.996	.998	.999	1.000		
7.0	.901	.947	.973	.987	.994	.998	.999	1.000		
7.2	.887	.937	.967	.984	.993	.997	.999	.999	1.000	
7.4	.871	.926	.961	.980	.991	.996	.998	.999	1.000	
7.6	.854	.915	.954	.976	.989	.995	.998	.999	1.000	
7.8	.835	.902	.945	.971	.986	.993	.997	.999	1.000	
8.0	.816	.888	.936	.966	.983	.992	.996	.998	.999	1.000
8.5	.763	.849	.909	.949	.973	.986	.993	.997	.999	.999
9.0	.706	.803	.876	.926	.959	.978	.989	.995	.998	.999
9.5	.645	.752	.836	.898	.940	.967	.982	.991	.996	.998
10.0	.583	.697	.792	.864	.917	.951	.973	.986	.993	.997

λ \ x	20	21	22
8.5	1.000		
9.0	1.000		
9.5	.999	1.000	
10.0	.998	.999	1.000

λ \ x	0	1	2	3	4	5	6	7	8	9
10.5	.000	.000	.002	.007	.021	.050	.102	.179	.279	.397
11.0	.000	.000	.001	.005	.015	.038	.079	.143	.232	.341
11.5	.000	.000	.001	.003	.011	.028	.060	.114	.191	.289
12.0	.000	.000	.001	.002	.008	.020	.046	.090	.155	.242
12.5	.000	.000	.000	.002	.005	.015	.035	.070	.125	.201
13.0	.000	.000	.000	.001	.004	.011	.026	.054	.100	.166
13.5	.000	.000	.000	.001	.003	.008	.019	.041	.079	.135
14.0	.000	.000	.000	.000	.002	.006	.014	.032	.062	.109
14.5	.000	.000	.000	.000	.001	.004	.010	.024	.048	.088
15.0	.000	.000	.000	.000	.001	.003	.008	.018	.037	.070

λ \ x	10	11	12	13	14	15	16	17	18	19
10.5	.521	.639	.742	.825	.888	.932	.960	.978	.988	.994
11.0	.460	.579	.689	.781	.854	.907	.944	.968	.982	.991
11.5	.402	.520	.633	.733	.815	.878	.924	.954	.974	.986
12.0	.347	.462	.576	.682	.772	.844	.899	.937	.963	.979
12.5	.297	.406	.519	628	.725	.806	.869	.916	.948	.969
13.0	.252	.353	.463	.573	.675	.764	.835	.890	.930	.957
13.5	.211	.304	.409	.518	.623	.718	.798	.861	.908	.942
14.0	.176	.260	.358	.464	.570	.669	.756	.827	.883	.923
14.5	.145	.220	.311	413	.518	.619	711	.790	.853	.901
15.0	.118	.185	.268	.363	.466	.568	.664	.749	.819	.875

λ \ x	20	21	22	23	24	25	26	27	28	29
10.5	.997	.999	.999	1.000						
11.0	.995	.998	.999	1.000						
11.5	.992	.996	.998	.999	1.000					
12.0	.988	.994	.997	.999	.999	1.000				
12.5	.983	.991	.995	.998	.999	.999	1.000			
13.0	.975	.986	.992	.996	.998	.999	1.000			
13.5	.965	.980	.989	.994	.997	.998	.999	1.000		
14.0	.952	.971	.983	.991	.995	.997	.999	.999	1.000	
14.5	.936	.960	.976	.986	.992	.996	.998	.999	.999	1.000
15.0	.917	.947	.967	.981	.989	.994	.997	.998	.999	1.000

λ \ x	4	5	6	7	8	9	10	11	12	13
16	.000	.001	.004	.010	.022	.043	.077	.127	.193	.275
17	.000	.001	.002	.005	.013	.026	.049	.085	.135	.201
18	.000	.000	.001	.003	.007	.015	.030	.055	.092	.143
19	.000	.000	.001	.002	.004	.009	.018	.035	.061	.098
20	.000	.000	.000	.001	.002	.005	.011	.021	.039	.066
21	.000	.000	.000	.000	.001	.003	.006	.013	.025	.043
22	.000	.000	.000	.000	.001	.002	.004	.008	.015	.028
23	.000	.000	.000	.000	.000	.001	.002	.004	.009	.017
24	.000	.000	.000	.000	.000	.000	001	.003	.005	.011
25	.000	.000	.000	.000	.000	.000	.001	.001	.003	.006
	14	15	16	17	18	19	20	21	22	23
16	.368	.467	.566	.659	.742	.812	.868	.911	.942	.963
17	.281	.371	.468	.564	.655	.736	.805	.861	.905	.937
18	.208	.287	.375	.469	.562	.651	.731	.799	.855	.899
19	.150	.215	.292	.378	469	.561	.647	.725	.793	.849
20	.105	.157	.221	.297	.381	.470	.559	.644	.721	.787
21	.072	.111	.163	.227	.302	.384	.471	.558	.640	.716
22	.048	.077	.117	.169	.232	.306	.387	.472	.556	.637
23	.031	.052	.082	.123	.175	.238	.310	.389	.472	.555
24	.020	.034	.056	.087	.128	.180	.243	.314	.392	.473
25	.012	.022	.038	.060	.092	.134	.185	.247	.318	.394
	24	25	26	27	28	29	30	31	32	33
16	.978	.987	.993	.996	.998	.999	.999	1.000		
17	.959	.975	.985	.991	.995	.997	.999	.999	1.000	
18	.932	.955	.972	.983	.990	.994	.997	.998	.999	1.000
19	.893	.927	.951	.969	.980	.988	.993	.996	.998	.999
20	.843	.888	.922	.948	.966	.978	.987	.992	.995	.997
21	.782	.838	.883	.917	.944	.963	.976	.985	.991	.994
22	.712	.777	.832	.877	.913	.940	.959	.973	.983	.989
23	.635	.708	.772	.827	.873	.908	.936	.956	.971	.981
24	.554	.632	.704	.763	.823	.868	.904	.932	.953	.969
25	.473	.553	.629	.700	.763	.818	.863	.900	.929	.950
	34	35	36	37	38	39	40	41	42	43
19	.999	1.000								
20	.999	.999	1.000							
21	.997	.998	.999	.999	1.000					
22	.994	.996	.998	.999	.999	1.000				
23	.988	.993	.996	.997	.999	.999	1.000			
24	.979	.987	.992	.995	.997	.998	.999	.999	1.000	
25	.966	.978	.985	.991	.994	.997	.998	.999	.999	1.000

TABLE 2.1a. $P(x|\lambda)$ FOR SMALL VALUES OF λ

[λ = 0005, 0.001 (0.001) 0.009]

$x \backslash \lambda$	0.0005	0.001	0.002	0.003	0.004
0	.9995001	9990005	9980020	.9970045	.9960080
1	.9999999	9999995	9999980	9999955	9999920
2	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

$x \backslash \lambda$	0.005	0.006	0.007	0.008	0.009
0	.9950125	.9940180	.9930244	.9920319	.9910404
1	.9999876	.9999821	.9999756	9999682	.9999598
2	1.0000000	1.0000000	.9999999	.9999999	.9999999
3			1.0000000	1.0000000	1.0000000

For small values of π and large values of n (for $\pi < 0.10$ and definitely for $n\pi < 5$) the binomial distribution is better approximated by the Poisson distribution than by the normal distribution. For example to find the probability of getting 5 or less defectives in a sample of 200 items from a process with fraction defective 0.02, we have $n\pi = 0.02 \times 200 = 4$. From Table 2.1 for $\lambda = 4$ and $x = 5$. the required probability is 0.785 which is close to the true value of 0.78672.

For large values of λ, the following normal approximation may be used.

$$P(x|\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x+0.5-\lambda)/\sqrt{\lambda}} e^{-t^2/2}\, dt.$$

For example to find $P(x|\lambda)$ for $\lambda = 25$ and $x = 27$, we have

$$\frac{x+0.5-\lambda}{\sqrt{\lambda}} = \frac{27+0.5-25}{5} = 0.5$$

From Table 3.1, the probability for normal deviate 0.5 is 0.500000+0.191462 = 0.691462, which is close to the true value of $P(x|\lambda) = 0.700$

There exists an exact relationship between cumulative Poisson probability and the probability integral of the χ^2 distribution, *i.e.*

$$P(x|\lambda) = 2^{-\nu/2}\{\Gamma(\nu/2)\}^{-1} \int_{2\lambda}^{\infty} e^{-\frac{1}{2}u}\, u^{\frac{\nu}{2}-1}\, du.$$

where $\nu = 2(x+1)$ for $x = 0, 2, \ldots\ldots$ etc.

2.2. Confidence Intervals For The Poisson Mean

a. Introduction

Table 2.2 gives two sided 95+% and 99+% confidence limits for the Poisson parameter λ (which is the mean of the Poisson distribution) based on a single observation x. Since the sum of n independent Poisson variables is also distributed according to the Poisson law with parameter $n\lambda$, we can find, by considering the sum of the observations as the variable, the confidence interval for $n\lambda$ and hence for λ, when there are n observations from the Poisson distribution.

The confidence intervals given in Table 2.2 follow the same principle as mentioned in **1.3a** and are based on tables provided by Crow and Gardner (1959).

The limits in Table 2.2 are given correct to two places of decimal, for values of $x = 0(1)50$. For higher values of x one may use the following limits derived from the normal approximation to the Poisson distribution

confidence coefficient	lower limit	upper limit
0.95	$x-1.96\sqrt{x}$	$x+1.96\sqrt{x}$
0.99	$x-2.58\sqrt{x}$	$x+2.58\sqrt{x}$

Example. A total number of 30 seeds were observed in a *sample* of $n = 20$ glass sheets manufactured by a certain process. It is required to find the 95% confidence interval for the process *average number* λ of seeds *per sheet*.

Entering Table 2.2 with $x = 30$ the 95% limits for $n\lambda$ ($n = 20$ in this example) are read as (20.33, 41.75). For these the 95% confidence limits for the process average number (λ) of seeds per sheet are given by

$$\left(\frac{20.33}{20}, \frac{41.75}{20}\right) \quad \text{or } (1.02,\ 2.09).$$

b. One sided confidence limits

With c as the observed value of x, the $100\alpha\%$ lower bound on λ is the smallest value of λ that satisfies the inequality

$$Q(c\,|\,\lambda) = \sum_{x=c}^{\infty} p(x\,|\,\lambda) \geqslant 1-\alpha.$$

Since

$$Q(c\,|\,\lambda) = \int_0^{\lambda} \frac{e^{-t}\,t^{c-1}}{\Gamma(c)}\,dt$$

the $100\alpha\%$ lower bound for λ is seen to coincide with half the value of the lower $100(1-\alpha)\%$ point of the chi-square distribution with $2c$ degree of freedom (Table 5.1).

Similarly the $100\alpha\%$ upper bound for λ is given by $U/2$ where U is the upper $100(1-\alpha)\%$ point of the chi-square distribution with $(2c+2)$ degrees of freedom.

Example. The upper 5% point of chisquare with 62 d.f. is 81.4. Hence with the same data as in the earlier example one may assert with 95% confidence that the average number of seeds per manufactured sheet does not exceed $\frac{1}{20}\times\frac{1}{2}\times 81.4 = 2.035$.

c. Tests of significance

Table 2.2 can be used for testing a simple hypothesis regarding λ when alternatives are both-sided. A hypothesis is rejected when the value of λ it specifies falls outside the confidence interval corresponding to the observed value of x.

Table 5 would similarly be useful for one sided tests on λ.

d. Some other tables

1. Crow, E. L. and Gardner, R. S. (1959) : Confidence Intervals for the Expectation of a Poisson Variable, *Biometrika*, Vol. **46**, pp. 441–453.
   80^+, 90^+, 95^+, 99^+, and 99.9^+% confidence intervals correct to two places of decimal, $x = 0(1)300$.

2. Pearson, E. S. and Hartley, H. O. (Eds.) (1957) : *Biometrika Tables for Statisticians*, Biometrika Trust, Cambridge University Press.
   Table 40 : 90^+, 95^+, 98^+, 99^+ and 99.8^+% confidence intervals, correct to two places of decimal, obtained from two sided tests with equal tail areas. $x = 0(1)30(5)50$.

TABLE 2.2. CONFIDENCE INTERVALS FOR THE POISSON MEAN

95^+% and 99^+% confidence coefficients

x	95% limits		99% limits	
0	0.000	3.285	0.000	4.771
1	0.051	5.323	0.010	6.914
2	0.355	6.686	0.149	8.727
3	0.818	8.102	0.436	10.473
4	1.366	9.598	0.823	12.347
5	1.970	11.177	1.279	13.793
6	2.613	12.817	1.785	15.277
7	3.285	13.765	2.330	16.801
8	3.285	14.921	2.906	18.362
9	4.460	16.768	3.507	19.462
10	5.323	17.633	4.130	20.676
11	5.323	19.050	4.771	22.042
12	6.686	20.335	4.771	23.765
13	6.686	21.364	5.829	24.925
14	8.102	22.945	6.668	25.992
15	8.102	23.762	6.914	27.718
16	9.598	25.400	7.756	28.852
17	9.598	26.306	8.727	29.900
18	11.177	27.735	8.727	31.839
19	11.177	28.966	10.009	32.547
20	12.817	30.017	10.473	34.183
21	12.817	31.67	11.242	35.204
22	13.765	32.277	12.347	36.544
23	14.921	34.048	12.347	37.819
24	14.921	34.665	13.793	38.939
25	16.768	36.030	13.793	40.373
26	16.77	37.67	15.28	41.39
27	17.63	38.16	15.28	42.85
28	19.05	39.76	16.80	43.91
29	19.05	40.94	16.80	45.26
30	20.33	41.75	18.36	46.50
31	21.36	43.45	18.36	47.62
32	21.36	44.26	19.46	49.13
33	22.94	45.28	20.28	49.96
34	23.76	47.02	20.68	51.78
35	23.76	47.69	22.04	52.28
36	25.40	48.74	22.04	54.03
37	26.31	50.42	23.76	54.74
38	26.31	51.29	23.76	56.14
39	27.73	52.15	24.92	57.61
40	28.97	53.72	25.83	58.35
41	28.97	54.99	25.99	60.39
42	30.02	55.51	27.72	60.59
43	31.67	56.99	27.72	62.13
44	31.67	58.72	28.85	63.63
45	32.28	58.84	29.90	64.26
46	34.05	60.24	29.90	65.96
47	34.66	61.90	31.84	66.81
48	34.66	62.81	31.84	67.92
49	36.03	63.49	32.55	69.83
50	37.67	64.95	34.18	70.05

3. THE STANDARD NORMAL DISTRIBUTION

3.1. Ordinates and Probability Integral

a. Introduction

Table 3.1 provides, correct to six places of decimal, values of the ordinates of the standard normal distribution

$$N(x) = N(x\,|\,0{,}1) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2},\ x = 0(0.01)\,3\,(0.1)\,4$$

and the values of the probability integral

$$P(x) = \int_0^x N(w)dw \text{ for } x = 0(0.001)\,3\,(0.01)\;(0.1)\,4.9.$$

From symmetry $N(x) = N(-x)$ and for non-negative numbers a and b, $(a < b)$

$$\int_a^b N(w)dw = P(b)-P(a) = \int_{-b}^{-a} N(w)dw$$

$$\int_{-a}^b N(w)dw = P(b)+P(a) = \int_{-b}^{a} N(w)dw$$

Example. The score S in a certain test is known to be normally distributed with mean 50 and standard deviation 10. Determine the proportion of cases for which the scores lie between (i) 35 and 55, and (ii) 55 and 67.

The distribution of $w = (S-50)/10$ is standard normal. Hence for (i) the answer is $\int_{-1.5}^{0.5} N(w)dw = P(0.5)+P(1.5) = 0.191462+0.433193 = 0.624655$.

Similarly the answer for (ii) is

$$\int_{0.5}^{1.7} N(w)dw = P(1.7)-P(0.5) = 0.455435-0.191462 = 0.263973.$$

b. Derivatives of N(x)

The Tchebycheff-Hermite polynomials $H_r(x)$ are defined by equations

$$\frac{d^rN(x)}{dx^r} = (-1)^r\, H_r(x)N(x)$$

$$H_r(x)=x^r-\binom{r}{2}x^{r-2}+1\times3\binom{r}{4}x^{r-4}-1\times3\times5\binom{r}{6}x^{r-6}+1\times3\times5\times7\binom{r}{8}x^{r-8}-$$

The table below gives the coefficients in $H_r(x)$ for r upto 10

COEFFICIENTS IN HERMITE POLYNOMIALS

r	x	x^3	x^5	x^7	x^9
1	1				
3	−3	1			
5	15	−10	1		
7	−105	105	−21	1	
9	945	−1260	378	−36	1

r	x^0	x^2	x^4	x^6	x^8	x^{10}
2	−1	1				
4	3	−6	1			
6	−15	45	−15	1		
8	105	−420	210	−28		
10	−945	4725	−3150	630	−45	1

c. Direct interpolation in Table 3.1

Formulae for interpolation are derived from the following Taylor expansions:

$$N(x) = N(x_0)\left[1-aH_1(x_0)+\frac{a^2}{2}H_2(x_0)-\frac{a^3}{6}H_3(x_0)+\ldots\right]$$

$$= N(x_0)\left[1-ax_0+\frac{a^2(x_0^2-1)}{2}-\frac{a^3(x_0^3-3x_0)}{6}+\ldots\right]$$

$$P(x) = P(x_0)+N(x_0)\left[a-\frac{a^2}{2}H_1(x_0)+\frac{a^3}{6}H_2(x_0)-\ldots\right]$$

$$= P(x_0)+N(x_0)\left[a-\frac{a^2x_0}{2}+\frac{a^3(x_0^2-1)}{6}-\ldots\right]$$

where x_0 denotes the tabular argument nearest to x for which answer is required and $a = x-x_0$.

For $N(x)$, the maximum error in using upto linear terms (linear in a) is $0.1995a^2$ and upto quadratic terms is $0.0918a^3$. For $P(x)$ the maximum error in using upto linear terms only is $0.1210a^2$ and upto quadratic terms, is $0.0665a^3$.

Example 1. Determine $N(0.0149)$

Choosing $x_0 = 0.01$, we have $a = 0.0049$. Then

$$N(.0149) = N(x_0)\left[1-ax_0+\frac{a^2(x_0^2-1)}{2}\right]$$

$$= 0.398922(1-0.000049-0.000012) = 0.398898 \text{ (to 6 decimal places)}$$

Example 2. Determine $P(1.0236)$

We use a slightly different formula for interpolation of $P(x)$,

$$P(x) = P(x_0)+N(x_0^*)\left[a-\frac{a^2x}{2}\right]$$

where x_0 is the tabular argument closest to x and x_0^* is x_0 rounded to two places of decimals. The substitution of $N(x_0^*)$ for $N(x_0)$ in the original formula does not introduce any serious error and the accuracy of this formula is comparable to the one considered earlier. Choosing $x_0 = 1.024$, we have $a = -0.0004$, and $x_0^* \doteq 1.02$.

Then
$$P(1.0236) = 0.349432 + 0.237132 \times [-0.0004]$$
$$= 0.349432 - 0.000095 = 0.349337 \text{ (to 6 places).}$$

d. Inverse interpolation

Suppose it is required to find x corresponding to a given value of $P(x) = A$, between two consecutive tabular entries in Table 3.1. Let x_0 be the argument corresponding to the nearest entry. The following formula determines x correct to five places of decimal for $x \leqslant 1.1663$ and at least to four decimal places elsewhere :

$$x = x_0 + \frac{A - P(x_0)}{N(x_0)}.$$

Example 3. Determine x for which $P(x) = 0.25$.

As in the formula for $P(x)$ in example 2, the above formula can be rewritten as

$$x = x_0 + \frac{A - P(x_0)}{N(x_0^*)}.$$

Choosing $x_0 = 0.674$, we have $x_0^* = 0.67$. Then $x = 0.674 + \dfrac{.000156}{0.318737} = 0.674 + 0.0049$ $= 0.67449$ (to 5 decimal places).

e. Some other tables

1. [U.S.] National Bureau of Standards (1953) : *Tables of Normal Probability Functions*, Applied Mathematics Series 23, Washington

   Table I gives $N(x)$ and $\int_{-x}^{x} N(w)dw$ correct to 15 places of decimal for $x = 0(0.0001)1(0.001)$ 7.800 (various) 8.285. Table II gives $N(x)$ and $\int_{-x}^{x} N(w)dw$ correct to 7 significant figures 6(0.01) 10.

2. Harvard University Computation Laboratory (1952) : *Tables of the Error Function and First Twenty Derivatives*. The Annals of the Computation Laboratory of Harvard University, Harvard Univ. Press, Cambridge (Massachussetts).

   The contents are as follows :

$\int_0^x N(w)dw$	6 dec	0(0.004) 4.892
$N(x)$	6 dec	0(0.004) 5.216
n-th derivative $D^n N(x)$:—		
$n = 1(1)4$	6 dec	0(0.004) 6.468
$n = 5(1)10$	6 dec	0(0.004) 8.236
$n = 11(1)15$	7 fig	0(0.002) 6.198
	and 6 dec	6.2(0.002) 9.61
$n = 16(1)20$	7 fig	0(0.002) 8.398
	and 6 dec	8.4(0.002)10.902.

7

TABLE 3.1. THE STANDARD NORMAL DISTRIBUTION: ORDINATES AND PROBABILITY INTEGRAL

[x = 0.00(0.01)0.34 for $N(x)$] [x = 0.000 (0.001) 0.349 for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.398942	0.00	.000000	.000399	.000798	.001197	.001596	.001995	.002394	.002793	.003192	.003590
.398922	0.01	.003989	.004388	.004787	.005186	.005585	.005984	.006383	.006782	.007181	.007579
.398862	0.02	.007978	.008377	.008776	.009175	.009574	.009973	.010371	.010770	.011169	.011568
.398763	0.03	.011966	.012365	.012764	.013163	.013561	.013960	.014359	.014757	.015156	.015555
.398623	0.04	.015953	.016352	.016751	.017149	.017548	.017946	.018345	.018743	.019142	.019540
.398444	0.05	.019939	.020337	.020736	.021134	.021532	.021931	.022329	.022727	.023126	.023524
.398225	0.06	.023922	.024320	.024719	.025117	.025515	.025913	.026311	.026709	.027107	.027505
.397966	0.07	.027903	.028301	.028699	.029097	.029495	.029893	.030290	.030688	.031086	.031484
.397668	0.08	.031881	.032279	.032677	.033074	.033472	.033869	.034267	.034664	.035062	.035459
.397330	0.09	.035856	.036254	.036651	.037048	.037445	.037843	.038240	.038637	.039034	.039431
.396953	0.10	.039828	.040225	.040622	.041019	.041415	.041812	.042209	.042606	.043002	.043399
.396536	0.11	.043795	.044192	.044588	.044985	.045381	.045777	.046174	.046570	.046966	.047362
.396080	0.12	.047758	.048154	.048550	.048946	.049342	.049738	.050134	.050530	.050926	.051321
.395585	0.13	.051717	.052112	.052508	.052903	.053299	.053694	.054089	.054485	.054880	.055275
.395052	0.14	.055670	.056065	.056460	.056855	.057250	.057645	.058039	.058434	.058829	.059223
.394479	0.15	.059618	.060012	.060407	.060801	.061195	.061589	.061983	.062378	.062772	.063166
.393868	0.16	.063559	.063953	.064347	.064741	.065134	.065528	.065922	.066315	.066708	.067102
.393219	0.17	.067495	.067888	.068281	.068674	.069067	.069460	.069853	.070246	.070639	.071031
.392531	0.18	.071424	.071816	.072209	.072601	.072993	.073385	.073778	.074170	.074562	.074954
.391806	0.19	.075345	.075737	.076129	.076521	.076912	.077304	.077695	.078086	.078477	.078869
.391043	0.20	.079260	.079651	.080042	.080432	.080823	.081214	.081605	.081995	.082386	.082776
.390242	0.21	.083166	.083556	.083946	.084337	.084726	.085116	.085506	.085896	.086285	.086675
.389404	0.22	.087064	.087454	.087843	.088232	.088621	.089010	.089399	.089788	.090177	.090566
.388529	0.23	.090954	.091343	.091731	.092119	.092508	.092896	.093284	.093672	.094059	.094447
.387617	0.24	.094835	.095222	.095610	.095997	.096385	.096772	.097159	.097546	.097933	.098320
.386668	0.25	.098706	.099093	.099479	.099866	.100252	.100638	.101025	.101411	.101797	.102182
.385683	0.26	.102568	.102954	.103339	.103725	.104110	.104495	.104880	.105265	.105650	.106035
.384663	0.27	.106420	.106804	.107189	.107573	.107958	.108342	.108726	.109110	.109494	.109878
.383606	0.28	.110261	.110645	.111028	.111412	.111795	.112178	.112561	.112944	.113327	.113709
.382515	0.29	.114092	.114474	.114857	.115239	.115621	.116003	.116385	.116767	.116148	.117530
.381388	0.30	.117911	.118293	.118674	.119055	.119436	.119817	.120198	.120578	.120959	.121339
.380226	0.31	.121720	.122100	.122480	.122860	.123239	.123619	.123999	.124378	.124758	.125137
.379031	0.32	.125516	.125895	.126274	.126652	.127031	.127409	.127788	.128166	.128544	.128922
.377801	0.33	.129300	.129678	.130055	.130433	.130810	.131187	.131565	.131942	.132318	.132695
.376537	0.34	.133072	.133448	.133825	.134201	.134577	.134953	.135329	.135704	.136080	.136455

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION: ORDINATES AND PROBABILITY INTEGRAL

[$x = 0.35(0.01)0.69$ for $N(x)$] [$x = 0.350(0.001)0.699$ for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.375240	0.35	.136831	.137206	.137581	.137956	.138331	.138705	.139080	.139454	.139828	.140202
.373911	0.36	.140576	.140950	.141324	.141698	.142071	.142444	.142817	.143190	.143563	.143936
.372548	0.37	.144309	.144681	.145054	.145426	.145798	.146170	.146542	.146913	.147285	.147656
.371154	0.38	.148027	.148398	.148769	.149140	.149511	.149881	.150252	.150622	.150992	.151362
.369728	0.39	.151732	.152101	.152471	.152840	.153209	.153579	.153947	.154316	.154685	.155053
.368270	0.40	.155422	.155790	.156158	.156526	.156894	.157261	.157629	.157996	.158363	.158730
.366782	0.41	.159097	.159464	.159830	.160197	.160563	.160929	.161295	.161661	.162026	.162392
.365263	0.42	.162757	.163122	.163487	.163852	.164217	.164582	.164946	.165310	.165674	.166038
.363714	0.43	.166402	.166766	.167129	.167493	.167856	.168219	.168582	.168944	.169307	.169669
.362135	0.44	.170031	.170394	.170755	.171117	.171479	.171840	.172201	.172562	.172923	.173284
.360527	0.45	.173645	.174005	.174366	.174726	.175086	.175445	.175805	.176164	.176524	.176883
.358890	0.46	.177242	.177601	.177959	.178318	.178676	.179034	.179392	.179750	.180108	.180465
.357225	0.47	.180822	.181180	.181537	.181893	.182250	.182607	.182963	.183319	.183675	.184031
.355533	0.48	.184386	.184742	.185097	.185452	.185807	.186162	.186516	.186871	.187225	.187579
.353812	0.49	.187933	.188287	.188640	.188994	.189347	.189700	.190053	.190405	.190758	.191110
.352065	0.50	.191462	.191814	.192166	.192518	.192869	.193221	.193572	.193923	.194273	.194624
.350292	0.51	.194974	.195324	.195674	.196024	.196374	.196723	.197073	.197422	.197771	.198120
.348493	0.52	.198468	.198817	.199165	.199513	.199861	.200208	.200556	.200903	.201250	.201597
.346668	0.53	.201944	.202291	.202637	.202983	.203329	.203675	.204021	.204366	.204711	.205057
.344818	0.54	.205401	.205746	.206091	.206435	.206779	.207123	.207467	.207811	.208154	.208497
.342944	0.55	.208840	.209183	.209526	.209868	.210211	.210553	.210895	.211236	.211578	.211919
.341046	0.56	.212260	.212601	.212942	.213283	.213623	.213963	.214303	.214643	.214983	.215322
.339124	0.57	.215661	.216000	.216339	.216678	.217016	.217354	.217692	.218030	.218368	.218705
.337180	0.58	.219043	.219380	.219717	.220053	.220390	.220726	.221062	.221398	.221734	.222069
.335213	0.59	.222405	.222740	.223075	.223409	.223744	.224078	.224412	.224746	.225080	.225414
.333225	0.60	.225747	.226080	.226413	.226746	.227078	.227411	.227743	.228075	.228406	.228738
.331215	0.61	.229069	.229400	.229731	.230062	.230392	.230723	.231053	.231383	.231712	.232042
.329184	0.62	.232371	.232700	.233029	.233358	.233686	.234014	.234343	.234670	.234998	.235325
.327133	0.63	.235653	.235980	.236307	.236633	.236960	.237286	.237612	.237938	.238263	.238589
.325062	0.64	.238914	.239239	.239563	.239888	.240212	.240536	.240860	.241184	.241508	.241831
.322972	0.65	.242154	.242477	.242799	.243122	.243444	.243766	.244088	.244410	.244731	.245052
.320864	0.66	.245373	.245694	.246014	.246335	.246655	.246975	.247294	.247614	.247933	.248252
.318737	0.67	.248571	.248890	.249208	.249526	.249844	.250162	.250480	.250797	.251114	.251431
.316593	0.68	.251748	.252064	.252381	.252697	.253012	.253328	.253643	.253959	.254274	.254588
.314432	0.69	.254903	.255217	.255531	.255845	.256159	.256472	.256786	.257099	.257411	.257724

TABLE 3.1 (*continued*). THE STANDARD NORMAL DISTRIBUTION : ORDINATES AND PROBABILITY INTEGRAL

[$x = 0.70(0.01)1.04$ for $N(x)$] [$x = 0.700(0.001)1.049$ for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.312254	0.70	.258036	.258348	.258660	.258972	.259284	.259595	.259906	.260217	.260527	.260838
.310060	0.71	.261148	.261458	.261768	.262077	.262386	.262695	.263004	.263313	.263621	.263930
.307851	0.72	.264238	.264545	.264853	.265160	.265467	.265774	.266081	.266387	.266693	.266999
.305627	0.73	.267305	.267610	.267916	.268221	.268526	.268830	.269135	.269439	.269743	.270047
.303389	0.74	.270350	.270653	.270956	.271259	.271562	.271864	.272166	.272468	.272770	.273071
.301137	0.75	.273373	.273674	.273974	.274275	.274575	.274876	.275175	.275475	.275775	.276074
.298872	0.76	.276373	.276671	.276970	.277268	.277566	.277864	.278162	.278459	.278756	.279053
.296595	0.77	.279350	.279647	.279943	.280239	.280535	.280830	.281126	.281421	.281715	.282010
.294305	0.78	.282305	.282599	.282893	.283186	.283480	.283773	.284066	.284359	.284652	.284944
.292004	0.79	.285236	.285528	.285820	.286111	.286402	.286693	.286984	.287274	.287565	.287855
.289692	0.80	.288145	.288434	.288724	.289013	.289302	.289590	.289879	.290167	.290455	.290742
.287369	0.81	.291030	.291317	.291604	.291891	.292178	.292464	.292750	.293036	.293321	.293607
.285036	0.82	.293892	.294177	.294462	.294746	.295030	.295314	.295598	.295881	.296165	.296448
.282694	0.83	.296731	.297013	.297296	.297578	.297860	.298141	.298423	.298704	.298985	.299265
.280344	0.84	.299546	.299826	.300106	.300386	.300665	.300945	.301224	.301502	.301781	.302059
.277985	0.85	.302337	.302615	.302893	.303170	.303448	.303724	.304001	.304278	.304554	.304830
.275618	0.86	.305105	.305381	.305656	.305931	.306206	.306481	.306755	.307029	.307303	.307576
.273244	0.87	.307850	.308123	.308396	.308668	.308941	.309213	.309485	.309757	.310028	.310299
.270864	0.88	.310570	.310841	.311112	.311382	.311652	.311922	.312191	.312461	.312730	.312998
.268477	0.89	.313267	.313535	.313804	.314071	.314339	.314606	.314874	.315141	.315407	.315674
.266085	0.90	.315940	.316206	.316472	.316737	.317002	.317267	.317532	.317797	.318061	.318325
.263688	0.91	.318589	.318852	.319116	.319379	.319642	.319904	.320167	.320429	.320691	.320952
.261286	0.92	.321214	.321475	.321736	.321996	.322257	.322517	.322777	.323037	.323296	.323555
.258881	0.93	.323814	.324073	.324332	.324590	.324848	.325106	.325363	.325621	.325878	.326135
.256471	0.94	.326391	.326648	.326904	.327160	.327415	.327671	.327926	.328181	.328435	.328690
.254059	0.95	.328944	.329198	.329452	.329705	.329958	.330211	.330464	.330716	.330969	.331221
.251644	0.96	.331472	.331724	.331975	.332226	.332477	.332728	.332978	.333228	.333478	.333727
.249228	0.97	.333977	.334226	.334475	.334723	.334972	.335220	.335468	.335715	.335963	.336210
.246809	0.98	.336457	.336704	.336950	.337196	.337442	.337688	.337933	.338179	.338424	.338668
.244390	0.99	.338913	.339157	.339401	.339645	.339889	.340132	.340375	.340618	.340860	.341103
.241971	1.00	.341345	.341587	.341828	.342070	.342311	.342552	.342792	.343033	.343273	.343513
.239551	1.01	.343752	.343992	.344231	.344470	.344709	.344947	.345185	.345423	.345661	.345899
.237132	1.02	.346136	.346373	.346610	.346846	.347082	.347318	.347554	.347790	.348025	.348260
.234714	1.03	.348495	.348730	.348964	.349198	.349432	.349666	.349899	.350132	.350365	.350593
.232297	1.04	.350830	.351062	.351294	.351526	.351757	.351989	.352219	.352450	.352681	.352911

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION : ORDINATES AND PROBABILITY INTEGRAL

[x = 1.05(0.01)1.39 for $N(x)$] [x = 1.050(0.001)1.399 for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.229882	1.05	.353141	.353371	.353600	.353830	.354059	.354287	.354516	.354744	.354972	.355200
.227470	1.06	.355428	.355655	.355882	.356109	.356336	.356562	.356788	.357014	.357240	.357465
.225060	1.07	.357690	.357915	.358140	.358364	.358589	.358813	.359036	.359260	.359483	.359706
.222653	1.08	.359929	.360151	.360374	.360596	.360818	.361039	.361261	.361482	.361702	.361923
.220251	1.09	.362143	.362364	.362583	.362803	.363023	.363242	.363461	.363679	.363898	.364116
.217852	1.10	.364334	.364552	.364769	.364986	.365203	.365420	.365637	.365853	.366069	.366285
.215458	1.11	.366500	.366716	.366931	.367146	.367360	.367575	.367789	.368003	.368217	.368430
.213069	1.12	.368643	.368856	.369069	.369281	.369493	.369705	.369917	.370129	.370340	.370551
.210686	1.13	.370762	.370972	.371183	.371393	.371603	.371812	.372022	.372231	.372440	.372648
.208308	1.14	.372857	.373065	.373273	.373481	.373688	.373895	.374102	.374309	.374516	.374722
.205936	1.15	.374928	.375134	.375339	.375545	.375750	.375955	.376159	.376364	.376568	.376772
.203571	1.16	.376976	.377179	.377382	.377585	.377788	.377991	.378193	.378395	.378597	.378798
.201214	1.17	.379000	.379201	.379401	.379602	.379802	.380003	.380203	.380402	.380602	.380801
.198863	1.18	.381000	.381199	.381397	.381595	.381793	.381991	.382189	.382386	.382583	.382780
.196520	1.19	.382977	.383173	.383369	.383565	.383761	.383956	.384152	.384347	.384541	.384736
.194186	1.20	.384930	.385124	.385318	.385512	.385705	.385898	.386091	.386284	.386476	.386669
.191860	1.21	.386861	.387052	.387244	.387435	.387626	.387817	.388008	.388198	.388388	.388578
.189543	1.22	.388768	.388957	.389146	.389335	.389524	.389712	.389901	.390089	.390277	.390464
.187235	1.23	.390651	.390839	.391025	.391212	.391399	.391585	.391771	.391956	.392142	.392327
.184937	1.24	.392512	.392697	.392882	.393066	.393250	.393434	.393618	.393801	.393984	.394167
.182649	1.25	.394350	.394533	.394715	.394897	.395079	.395261	.395442	.395623	.395804	.395985
.180371	1.26	.396165	.396346	.396526	.396705	.396885	.397064	.397243	.397422	.397601	.397779
.178104	1.27	.397958	.398136	.398313	.398491	.398668	.398845	.399022	.399199	.399375	.399551
.175847	1.28	.399727	.399903	.400079	.400254	.400429	.400604	.400778	.400953	.401127	.401301
.173602	1.29	.401475	.401648	.401821	.401994	.402167	.402340	.402512	.402684	.402856	.403028
.171369	1.30	.403200	.403371	.403542	.403713	.403883	.404054	.404224	.404394	.404563	.404733
.169147	1.31	.404902	.405071	.405240	.405409	.405577	.405745	.405913	.406081	.406248	.406415
.166937	1.32	.406582	.406749	.406916	.407082	.407248	.407414	.407580	.407746	.407911	.408076
.164740	1.33	.408241	.408405	.408570	.408734	.408898	.409062	.409225	.409389	.409552	.409715
.162555	1.34	.409877	.410040	.410202	.410364	.410526	.410687	.410849	.411010	.411171	.411332
.160383	1.35	.411492	.411652	.411812	.411972	.412132	.412291	.412450	.412609	.412768	.412927
.158225	1.36	.413085	.413243	.413401	.413559	.413716	.413873	.414031	.414187	.414344	.414500
.156080	1.37	.414657	.414813	.414968	.415124	.415279	.415434	.415589	.415744	.415898	.416053
.153948	1.38	.416207	.416361	.416514	.416668	.416821	.416974	.417127	.417279	.417431	.417584
.151831	1.39	.417736	.417887	.418039	.418190	.418341	.418492	.418643	.418793	.418943	.419094

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION : ORDINATES AND PROBABILITY INTEGRAL

[$x = 1.40(0.01)1.74$ for $N(x)$] [$x = 1.400(0.001)1.749$ for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.149727	1.40	.419243	.419393	.419542	.419692	.419841	.419989	.420138	.420286	.420434	.420582
.147639	1.41	.420730	.420878	.421025	.421172	.421319	.421466	.421612	.421759	.421905	.422050
.145564	1.42	.422196	.422342	.422487	.422632	.422777	.422921	.423066	.423210	.423354	.423498
.143505	1.43	.423641	.423785	.423928	.424071	.424214	.424356	.424499	.424641	.424783	.424925
.141460	1.44	.425066	.425208	.425349	.425490	.425631	.425771	.425911	.426052	.426191	.426331
.139431	1.45	.426471	.426610	.426749	.426888	.427027	.427165	.427304	.427442	.427580	.427717
.137417	1.46	.427855	.427992	.428129	.428266	.428403	.428540	.428676	.428812	.428948	.429084
.135418	1.47	.429219	.429354	.429490	.429624	.429759	.429894	.430028	.430162	.430296	.430430
.133435	1.48	.430563	.430697	.430830	.430963	.431096	.431228	.431360	.431493	.431625	.431756
.131468	1.49	.431888	.432019	.432150	.432281	.432412	.432543	.432673	.432803	.432933	.433063
.129518	1.50	.433193	.433322	.433451	.433580	.433709	.433838	.433966	.434095	.434223	.434351
.127583	1.51	.434478	.434606	.434733	.434860	.434987	.435114	.435240	.435367	.435493	.435619
.125665	1.52	.435745	.435870	.435995	.436121	.436246	.436370	.436495	.436619	.436744	.436868
.123763	1.53	.436992	.437115	.437239	.437362	.437485	.437608	.437731	.437853	.437976	.438098
.121878	1.54	.438220	.438342	.438463	.438585	.438706	.438827	.438948	.439068	.439189	.439309
.120009	1.55	.439429	.439549	.439669	.439788	.439908	.440027	.440146	.440265	.440383	.440502
.118157	1.56	.440620	.440738	.440856	.440974	.441091	.441209	.441326	.441443	.441559	.441676
.116323	1.57	.441792	.441909	.442025	.442141	.442256	.442372	.442487	.442602	.442717	.442832
.114505	1.58	.442947	.443061	.443175	.443289	.443403	.443517	.443630	.443744	.443857	.443970
.112704	1.59	.444083	.444195	.444308	.444420	.444532	.444644	.444756	.444867	.444979	.445090
.110921	1.60	.445201	.445312	.445422	.445533	.445643	.445753	.445863	.445973	.446082	.446192
.109155	1.61	.446301	.446410	.446519	.446628	.446736	.446845	.446953	.447061	.447169	.447276
.107406	1.62	.447384	.447491	.447598	.447705	.447812	.447919	.448025	.448131	.448238	.448343
.105675	1.63	.448449	.448555	.448660	.448766	.448871	.448975	.449080	.449185	.449289	.449393
.103961	1.64	.449497	.449601	.449705	.449809	.449912	.450015	.450118	.450221	.450324	.450426
.102265	1.65	.450529	.450631	.450733	.450835	.450936	.451038	.451139	.451240	.451341	.451442
.100586	1.66	.451543	.451643	.451744	.451844	.451944	.452044	.452143	.452243	.452342	.452441
.098925	1.67	.452540	.452639	.452738	.452836	.452935	.453033	.453131	.453229	.453326	.453424
.097282	1.68	.453521	.453619	.453716	.453812	.453909	.454006	.454102	.454198	.454294	.454390
.095657	1.69	.454486	.454582	.454677	.454772	.454867	.454962	.455057	.455152	.455246	.455340
.094049	1.70	.455435	.455529	.455622	.455716	.455809	.455903	.455996	.456089	.456182	.456275
.092459	1.71	.456367	.456459	.456552	.456644	.456736	.456827	.456919	.457010	.457102	.457193
.090887	1.72	.457284	.457375	.457465	.457556	.457646	.457736	.457826	.457916	.458006	.458095
.089333	1.73	.458185	.458274	.458363	.458452	.458541	.458630	.458718	.458806	.458895	.458983
.087796	1.74	.459070	.459158	.459246	.459333	.459420	.459508	.459595	.459681	.459768	.459854

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION : ORDINATES AND PROBABILITY INTEGRAL

[x = 1.75(0.01)2.09 for $N(x)$] [x = 1.750(0.001)2.099 for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.086277	1.75	.459941	.460027	.460113	.460199	.460285	.460370	.460456	.460541	.460626	.460711
.084776	1.76	.460796	.460881	.460965	.461050	.461134	.461218	.461302	.461386	.461470	.461553
.083293	1.77	.461636	.461720	.461803	.461886	.461968	.462051	.462134	.462216	.462298	.462380
.081828	1.78	.462462	.462544	.462625	.462707	.462788	.462869	.462950	.463031	.463112	.463193
.080380	1.79	.463273	.463353	.463434	.463514	.463593	.463673	.463753	.463832	.463911	.463991
.078950	1.80	.464070	.464149	.464227	.464306	.464384	.464463	.464541	.464619	.464697	.464774
.077538	1.81	.464852	.464930	.465007	.465084	.465161	.465238	.465315	.465391	.465468	.465544
.076143	1.82	.465620	.465697	.465773	.465848	.465924	.465999	.466075	.466150	.466225	.466300
.074766	1.83	.466375	.466450	.466524	.466599	.466673	.466747	.466821	.466895	.466969	.467042
.073407	1.84	.467116	.467189	.467262	.467335	.467408	.467481	.467554	.467626	.467699	.467771
.072065	1.85	.467843	.467915	.467987	.468059	.468130	.468202	.468273	.468344	.468415	.468486
.070740	1.86	.468557	.468628	.468698	.468769	.468839	.468909	.468979	.469049	.469119	.469189
.069433	1.87	.469258	.469327	.469397	.469466	.469535	.469604	.469672	.469741	.469809	.469878
.068144	1.88	.469946	.470014	.470082	.470150	.470218	.470285	.470353	.470420	.470487	.470554
.066871	1.89	.470621	.470688	.470755	.470821	.470887	.470954	.471020	.471086	.471152	.471218
.065616	1.90	.471283	.471349	.471414	.471480	.471545	.471610	.471675	.471740	.471804	.471869
.064378	1.91	.471933	.471998	.472062	.472126	.472190	.472254	.472317	.472381	.472444	.472508
.063157	1.92	.472571	.472634	.472697	.472760	.472823	.472885	.472948	.473010	.473072	.473135
.061952	1.93	.473197	.473258	.473320	.473382	.473443	.473505	.473566	.473627	.473688	.473749
.060765	1.94	.473810	.473871	.473931	.473992	.474052	.474113	.474173	.474233	.474293	.474352
.059595	1.95	.474412	.474471	.474531	.474590	.474649	.474708	.474767	.474826	.474885	.474944
.058441	1.96	.475002	.475060	.475119	.475177	.475235	.475293	.475351	.475408	.475466	.475523
.057304	1.97	.475581	.475638	.475695	.475752	.475809	.475866	.475923	.475979	.476036	.476092
.056183	1.98	.476148	.476204	.476260	.476316	.476372	.476428	.476483	.476539	.476594	.476649
.055079	1.99	.476705	.476760	.476814	.476869	.476924	.476979	.477033	.477087	.477142	.477196
.053991	2.00	.477250	.477304	.477358	.477411	.477465	.477518	.477572	.477625	.477678	.477731
.052919	2.01	.477784	.477837	.477890	.477943	.477995	.478048	.478100	.478152	.478204	.478256
.051864	2.02	.478308	.478360	.478412	.478463	.478515	.478566	.478618	.478669	.478720	.478771
.050824	2.03	.478822	.478873	.478923	.478974	.479024	.479075	.479125	.479175	.479225	.479275
.049800	2.04	.479325	.479375	.479424	.479474	.479523	.479573	.479622	.479671	.479720	.479769
.048792	2.05	.479818	.479867	.479915	.479964	.480012	.480060	.480109	.480157	.480205	.480253
.047800	2.06	.480301	.480348	.480396	.480444	.480491	.480538	.480586	.480633	.480680	.480727
.046823	2.07	.480774	.480821	.480867	.480914	.480960	.481007	.481053	.481099	.481145	.481191
.045861	2.08	.481237	.481283	.481329	.481374	.481420	.481465	.481511	.481556	.481601	.481646
.044915	2.09	.481691	.481736	.481781	.481825	.481870	.481915	.481959	.482003	.482047	.482092

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION: ORDINATES AND PROBABILITY INTEGRAL

[$x = 2.10(0.01)2.44$ for $N(x)$] [$x = 2.100(0.001)2.449$ for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.043984	2.10	.482136	.482180	.482223	.482267	.482311	.482354	.482398	.482441	.482485	.482528
.043067	2.11	.482571	.482614	.482657	.482700	.482742	.482785	.482828	.482870	.482912	.482955
.042166	2.12	.482997	.483039	.483081	.483123	.483165	.483207	.483248	.483290	.483331	.483373
.041280	2.13	.483414	.483455	.483497	.483538	.483579	.483619	.483660	.483701	.483742	.483782
.040408	2.14	.483823	.483863	.483903	.483943	.483984	.484024	.484064	.484103	.484143	.484183
.039550	2.15	.484222	.484262	.484301	.484341	.484380	.484419	.484458	.484497	.484536	.484575
.038707	2.16	.484614	.484652	.484691	.484729	.484768	.484806	.484844	.484883	.484921	.484959
.037878	2.17	.484997	.485034	.495072	.485110	.485147	.485185	.485222	.485260	.485297	.485334
.037063	2.18	.485371	.485408	.485445	.485482	.485519	.485556	.485592	.485629	.485665	.485702
.036262	2.19	.485738	.485774	.485810	.485846	.485882	.485918	.485954	.485990	.486025	.586061
.035475	2.20	.486097	.486132	.486167	.486203	.486238	.486273	.486308	.486343	.486378	.486413
.034701	2.21	.486447	.486482	.486517	.486551	.486586	.486620	.486654	.486688	.486723	.486757
.033941	2.22	.486791	.486825	.486858	.486892	.486926	.486959	.486993	.487026	.487060	.487093
.033194	2.23	.487126	.487159	.487193	.487226	.487258	.487291	.487324	.487357	.487389	.487422
.032460	2.24	.487455	.487487	.487519	.487552	.487584	.487616	.487648	.487680	.487712	.487744
.031740	2.25	.487776	.487807	.487839	.487870	.487902	.487933	.487965	.487996	.488027	.488058
.031032	2.26	.488089	.488120	.488151	.488182	.488213	.488244	.488274	.488305	.488335	.488366
.030337	2.27	.488396	.488427	.488457	.488487	.488517	.488547	.488577	.488607	.488637	.488666
.029655	2.28	.488696	.488726	.488755	.488785	.488814	.488844	.488873	.488902	.488931	.488960
.028985	2.29	.488989	.489018	.489047	.489076	.489105	.489133	.489162	.489191	.489219	.489248
.028327	2.30	.489276	.489304	.489332	.489361	.489389	.489417	.489445	.489473	.489500	.489528
.027682	2.31	.489556	.489584	.489611	.489639	.489666	.489694	.489721	.489748	.489775	.489802
.027048	2.32	.489830	.489857	.489884	.489910	.489937	.489964	.489991	.490017	.490044	.490070
.026426	2.33	.490097	.490123	.490150	.490176	.490202	.490228	.490254	.490280	.490306	.490332
.025817	2.34	.490358	.490384	.490410	.490435	.490461	.490486	.490512	.490537	.490563	.490588
.025218	2.35	.490613	.490638	.490664	.490689	.490714	.490739	.490764	.490788	.490813	.490838
.024631	2.36	.490863	.490887	.490912	.490936	.490961	.490985	.491009	.491034	.491058	.491082
.024056	2.37	.491106	.491130	.491154	.491178	.491202	.491226	.491249	.491273	.491297	.491320
.023491	2.38	.491344	.491367	.491391	491414	.491437	.491460	.491484	.491507	.491530	.491553
.022937	2.39	.491576	.491599	.491622	.491644	.491667	.491690	.491712	.491735	.491758	.491780
.022395	2.40	.491802	.491825	.491847	.491869	.491892	.491914	.491936	.491958	.491980	.492002
.021862	2.41	.492024	.492046	.492067	.492089	.492111	.492132	.492154	.492175	.492197	.492218
.021341	2.42	.492240	.492261	.492282	.492304	.492325	.492346	.492367	.492388	.492409	.492430
.020829	2.43	.492451	.492471	.492492	.492513	.492534	.492554	.492575	.492595	.492616	.492636
.020328	2.44	.492656	.492677	.492697	.492717	.492737	.492757	.492777	.492797	.492817	.492837

TABLE 3.1. (*continued*). THE STANDARD NORMAL DISTRIBUTION: ORDINATES AND PROBABILITY INTEGRAL

[$x = 2.45(0.01)2.79$ for $N(x)$] [$x = 2.450(0.001)2.799$ for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.019837	2.45	.492857	.492877	.492897	.492916	.492936	.492956	.492975	.492995	.493014	.493034
.019356	2.46	.493053	.493072	.493092	.493111	.493130	.493149	.493168	.493187	.493206	.493225
.018885	2.47	.493244	.493263	.493282	.493301	.493320	.493338	.493357	.493375	.493394	.493412
.018423	2.48	.493431	.493449	.493468	.493486	.493504	.493522	.493541	.493559	.493577	.493595
.017971	2.49	.493613	.493631	.493649	.493667	.493684	.493702	.493720	.493738	.493755	.493773
.017528	2.50	.493790	.493808	.493825	.493843	.493860	.493877	.493895	.493912	.493929	.493946
.017095	2.51	.493963	.493981	.493998	.494015	.494031	.494048	.494065	.494082	.494099	.494116
.016670	2.52	.494132	.494149	.494166	.494182	.494199	.494215	.494232	.494248	.494264	.494281
.016254	2.53	.494297	.494313	.494329	.494345	.494362	.494378	.494394	.494410	.494426	.494442
.015843	2.54	.494457	.494473	.494489	.494505	.494520	.494536	.494552	.494567	.494583	.494598
.015449	2.55	.494614	.494629	.494645	.494660	.494675	.494691	.494706	.494721	.494736	.494751
.015060	2.56	.494766	.494781	.494796	.494811	.494826	.494841	.494856	.494871	.494886	.494900
.014678	2.57	.494915	.494930	.494944	.494959	.494973	.494988	.495002	.495017	.495031	.495046
.014305	2.58	.495060	.495074	.495089	.495103	.495117	.495131	.495145	.495159	.495173	.495187
.013940	2.59	.495201	.495215	.495229	.495243	.495257	.495270	.495284	.495298	.495312	.495325
.013583	2.60	.495339	.495352	.495366	.495379	.495393	.495406	.495420	.495433	.495446	.495460
.013234	2.61	.495473	.495486	.495499	.495512	.495526	.495539	.495552	.495565	.495578	.495591
.012892	2.62	.495604	.495616	.405629	.495642	.495655	.495668	.495680	.495693	.495706	.495718
.012558	2.63	.495731	.495743	.495756	.495768	.495781	.495793	.495806	.495818	.495830	.495842
.012232	2.64	.495855	.495867	.495879	.495891	.495903	.495915	.495928	.495940	.495952	.495963
.011912	2.65	.495975	.495987	.495999	.496011	.496023	.496035	.496046	.496058	.496070	.496081
.011600	2.66	.496093	.496105	.496116	.496128	.496139	.496151	.496162	.496173	.496185	.496196
.011295	2.67	.496207	.496219	.496230	.496241	.496252	.496264	.496275	.496286	.496297	.496308
.010997	2.68	.496319	.496330	.496341	.496352	.496363	.496374	.496384	.496395	.496406	.406417
.010706	2.69	.496427	.496438	.496449	.496459	.496470	.496481	.496491	.496502	.496512	.496523
.010421	2.70	.496533	.496543	.496554	.496564	.496574	.496585	.496595	.496605	.496615	.496626
.010143	2.71	.496636	.496646	.496656	.496666	.496676	.496686	.496696	.496706	.496716	.496726
.009871	2.72	.496736	.496746	.496756	.496765	.496775	.496785	.496795	.496804	.496814	.496824
.009606	2.73	.496833	.496843	.496852	.496862	.496871	.496881	.496890	.496900	.496909	.496919
.009347	2.74	.496928	.496937	.496947	.496956	.496965	.496974	.496984	.496993	.497002	.497011
.009094	2.75	.497020	.497029	.497038	.497047	.497056	.497065	.497074	.497083	.497092	.497101
.008846	2.76	.497110	.497119	.497128	.497136	.497145	.497154	.497163	.497171	.497180	.497189
.088605	2.77	.497197	.497206	.497214	.497223	.497231	.497240	.497248	.497257	.497265	.497274
.008370	2.78	.497282	.497290	.497299	.497307	.497315	.497324	.497332	.497340	.497348	.497356
.008140	2.79	.497365	.497373	.497381	.497389	.497397	.497405	.497413	.497421	.497429	.497437

TABLE 3.1. *(continued)*. THE STANDARD NORMAL DISTRIBUTION : ORDINATES AND PROBABILITY INTEGRAL

[x = 2.80(0.01)3.0(0.1) 4 for $N(x)$] [x = 2.800 (0.001) 3.00(0.01) 4.0(0.1) 4.9 for $P(x)$]

ordinate $N(x)$	x	probability integral $P(x)$ 0	1	2	3	4	5	6	7	8	9
.007915	2.80	.497445	.497453	.497461	.497469	.497476	.497484	.497492	.497500	.497507	.497515
.007697	2.81	.497523	.497531	.497538	.497546	.497554	.497561	.497569	.497576	.497584	.497591
.007483	2.82	.497599	.497606	.497614	.497621	.497629	.497636	.497643	.497651	.497658	.497665
.007274	2.83	.497673	.497680	.497687	.497694	.497702	.497709	.497716	.497723	.497730	.497737
.007071	2.84	.497744	.497751	.497758	.497765	.497772	.497779	.497786	.497793	.497800	.497807
.006873	2.85	.497814	.497821	.497828	.497835	.497841	.497848	.497855	.497862	.497868	.497875
.006679	2.86	.497882	.497888	.497895	.497902	.497908	.497915	.497922	.497928	.497935	.497941
.006491	2.87	.497948	.497954	.497961	.497967	.497973	.497980	.497986	.497993	.497999	.498005
.006307	2.88	.498012	.498018	.498024	.498030	.498037	.498043	.498049	.498055	.498062	.498068
.006127	2.89	.498074	.498080	.498086	.498092	.498098	.498104	.498110	.498116	.498122	.498128
.005953	2.90	.498134	.498140	.498146	.498152	.498158	.498164	.498170	.498175	.498181	.498187
.005782	2.91	.498193	.498199	.498204	.498210	.498216	.498222	.498227	.498233	.498239	.498244
.005616	2.92	.498250	.498255	.498261	.498267	.498272	.498278	.498283	.498289	.498294	.498300
.005454	2.93	.498305	.498311	.498316	.498321	.498327	.498332	.498338	.498343	.498348	.498354
.005296	2.94	.498359	.498364	.498370	.498375	.498380	.498385	.498390	.498396	.498401	.498406
.005143	2.95	.498411	.498416	.498421	.498426	.498432	.498437	.498442	.498447	.498452	.498457
.004993	2.96	.498462	.498467	.498472	.498477	.498482	.498487	.498491	.498496	.498501	.498506
.004847	2.97	.498511	.498516	.498521	.498525	.498530	.498535	.498540	.498545	.498549	.498554
.004705	2.98	.498559	.498563	.498568	.498573	.498577	.498582	.498587	.498591	.498596	.498601
.004567	2.99	.498605	.498610	.498614	.498619	.498623	.498628	.498632	.498637	.498641	.498646
.004432	3.0*	.498650	.498694	.498736	.498777	.498817	.498856	.498893	.498930	.498965	.498999
.003267	3.1	.499032	.499065	.499096	.499126	.499155	.499184	.499211	.499238	.499264	.499289
.002384	3.2	.499313	.499336	.499359	.499381	.499402	.499423	.499443	.499462	.499481	.499499
.001723	3.3	.499517	.499534	.499550	.499566	.499581	.499596	.499610	.499624	.499638	.499651
.001232	3.4	.499663	.499675	.499687	.499698	.499709	.499720	.499730	.499740	.499749	.499758
.000873	3.5	.499767	.499776	.499784	.499792	.499800	.499807	.499815	.499822	.499828	.499835
.000612	3.6	.499841	.499847	.499853	.499858	.499864	.499869	.499874	.499879	.499883	.499888
.000425	3.7	.499892	.499896	.499900	.499904	.499908	.499912	.499915	.499918	.499922	.499925
.000292	3.8	.499928	.499931	.499933	.499936	.499938	.499941	.499943	.499946	.499948	.499950
.000199	3.9	.499952	.499954	.499956	.499958	.499959	.499961	.499963	.499964	.499966	.499967
.000134	4. *	.499968	.499979	.499987	.499991	.499995	.499997	.499998	.499999	.499999	.500000

* Note the change in the interval of tabulation.

3.2. Percentage Points

a. Introduction

For various values of p, Table 3.2 provides the upper $100p\%$ points of the absolute value of the standard normal variable, or more explicitly it gives the value of x satisfying the equation

$$p = \int_{x}^{\infty} N(w)dw + \int_{-\infty}^{-x} N(w)dw = 2\int_{x}^{\infty} N(w)dw$$

Since $\frac{p}{2} = \int_{x}^{\infty} N(w)dw$, the tabular values may also be interpreted as the upper $50p\%$ point of the standard normal variable. The lower $50p\%$ point can be obtained by prefixing a negative sign to the value of the upper $50p\%$ point. Thus reading against $p = .24$ in Table 3.2, the upper 12% point of the standard normal variable is obtained as 1.174987. The lower 12% point is therefore -1.174987.

Table 3.2 also provides a short table of p (the probability of an observation falling outside the range $-x$ to x) for the following values of x

$$x = 0.25,\ 0.5(0.5)\ 5.0.$$

b. Application

Table 3.2 is useful in tests of significance, particularly in large sample tests using standard errors (see Chapter IV in Part I) and together with Table 3.1, in a limited sense, for probit analysis. A further use is in Cornish-Fisher type expansions for the fractiles of other variables having asymptotically a standard normal distribution. For t, F and X^2 these expansions are provided in explanatory notes preceding the corresponding tables.

TABLE 3.2 THE STANDARD NORMAL DISTRIBUTION: PERCENTAGE POINTS OF ABSOLUTE VALUE

p[1]	0	1	2	3	4	5	6	7	8	9
.0	∞	2.575829	2.326348	2.170090	2.053749	1.959964	1.880794	1.811911	1.750686	1.695398
.1	1.644854	1.598193	1.554774	1.514102	1.475791	1.439531	1.405072	1.372204	1.340755	1.310579
.2	1.281552	1.253565	1.226528	1.200359	1.174987	1.150349	1.126391	1.103063	1.080319	1.058122
.3	1.036433	1.015222	.994458	.974114	.954165	.934589	.915365	.896473	.877896	.859617
.4	.841621	.823894	.806421	.789192	.772193	.755415	.738847	.722479	.706303	.690309
.5	.674490	.658838	.643345	.628006	.612813	.597760	.582842	.568051	.553385	.538836
.6	.524401	.510073	.495850	.481727	.467699	.453762	.439913	.426148	.412463	.398855
.7	.385320	.371856	.358459	.345126	.331853	.318639	.305481	.292375	.279319	.266311
.8	.253347	.240426	.227545	.214702	.201893	.189118	.176374	.163658	.150969	.138304
.9	.125661	.113039	.100434	.087845	.075270	.062707	.050154	.037608	.025069	.012533

p	..	.001	.000,1	.000,01	.000,001	.000,000,1	.000,000,01	.000,000,001
x	..	3.29053	3.89059	4.41717	4.89164	5.32672	5.73073	6.10941

x	0.25	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
p	.802587	.617075	.317311	.133614	.045500	.012419	.002700	.000465	.000063	.000007	.000001

(1): The first digit of p after the decimal point is given in the column and the second digit in the row.

4. THE t-DISTRIBUTION

a. Introduction

Table 4.1 gives the p-th fractile of the t-distribution, for degrees of freedom $\nu = 1(1)30,\ 40(20)100,\ \infty$, the values of p being :

0.6, 0.7, 0.75, 0.8, 0.85, 0.9, .95, 0.975, 0.99, 0.995, 0.999, 0.9995.

Fractiles for the following values of p can also be easily deduced from Table 4 1, by changing sign because of symmetry (about the origin) of the t-distribution :

p : 0.0005, 0.001, 0.005, 0.01, 0.025, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.4.

Example : To find the fractile of t for $\nu = 4$, $p = 0.05$.

The required fractile is -2.132 (2.132 being the 0.95-th fractile of t for 4 degrees of freedom).

The last six columns of Table 4.1 directly provide critical values of $|t|$ for two-sided tests at the 10%, 5% and 2%, 1%, 0.2% and 0.1% levels of significance respectively. They also give the critical values of t for upper tail tests at the significance levels of 5%, 2.5% and 1%, 0.5%, 0.1% and 0.05%. A negative sign prefixed to these values would provide the critical values for lower tail tests.

3. Computing the fractiles for other degrees of freedom

For higher values of ν Cornish-Fisher expansion of t_p (the p-th fractile of t with ν d.f.) may be used to determine its value to any desired accuracy

$$t_{p\nu} = x + \frac{1}{\nu}\left(\frac{x^3+x}{4}\right) + \frac{1}{\nu^2}\left(\frac{5x^5+16x^3+3x}{96}\right) + \frac{1}{\nu^3}\left(\frac{3x^7+19x^5+17x^3-15x}{384}\right)$$

$$+\frac{1}{\nu^4}\left(\frac{79x^9+776x^7+1482x^5-1920x^3-945x}{92160}\right)+\ldots$$

where x is the p-th fractile of the standard normal distribution.

Values of x (the first term) and the coefficients of $1/\nu$, $1/\nu^2$, etc. in the expansion, for the different values of p covered in Table 4.1 are shown below

COEFFICIENTS* IN THE CORNISH-FISHER EXPANSION

coef. of	value of p										
	.975	.995	.9995	.95	.99	.999	.6	.7	75	.8	.9
1	1.95996	2.57583	3.29053	1.64485	2.32635	3.09023	0.25335	0.52440	0.67449	0.84162	1.28155
$1/\nu$	2.37227	4.91655	9.72973	1.52377	3.72907	8.15013	0.06740	0.16715	0.24533	0.35944	0.84658
$1/\nu^2$	2.8225	8.8348	26.1330	1.4202	5.7197	19.6925	0.0107	0.0425	0.0795	0.1477	0.5709
$1/\nu^3$	2.556	12.144	53.169	0.983	6.719	36.154	−0.009	−0.012	−0.005	0.017	0.259
$1/\nu^4$	1.6	12.1	79.4	0.4	5.6	48.6	0	0	0	0	0.1

* Sufficient figures are retained to ensure accuracy in the fourth decimal place for $n > 30$.

The coefficients for $p = 0.85$ of 1, $1/\nu$, $1/\nu^2$, $1/\nu^3$ and $1/\nu^4$ are 1.03643, 0.53744, 0.28023, 0.078 and 0.0 respectively.

c. Applications

Some uses of Table 4.1 are illustrated

(i) *One sample problem—test and confidence interval*

Example : The mean and sample variance of hardness (Rockwell E) determined from a sample of 10 pieces of die-cast aluminium are :

$$\bar{x} = 68.5 \quad s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = 2.5.$$

Are these consistent with the hypothesis that the average hardness μ in respect of the manufacturing process is 70 ?

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} = -3.0, \text{ and } |t| = 3.0.$$

The 5% and 1% level values of $|t|$ (for a two-sided test) for 9 d.f. being 2.262 and 3.250 respectively, the hypothesis can be rejected at the 5% level. On the basis of the data a 95% confidence statement of the following kind can be made :

(a) μ does not exceed $\bar{x}+1.833 \dfrac{s}{\sqrt{10}} = 69.42$,

or (b) μ does not fall below $\bar{x}-1.833 \dfrac{s}{\sqrt{10}} = 67.58$,

or (c) μ lies between $\bar{x}-2.262 \dfrac{s}{\sqrt{10}} = 67.37$ and $\bar{x}+2.262 \dfrac{s}{\sqrt{10}} = 69.63$

where 10 under square root in the denominator is the sample size and 1.833, 2.262 are upper 5 % and two-sided 5 % values of t from Table 4.1 corresponding to $n-1$ (= 9) d.f.

(ii) *Two-sample problem*

Example : The impact strength readings in foot pounds in samples of sheets from two lots were summarised as follows :

Lot 1 : Sample size $n_1 = 8$,

$$\bar{x}_1 = 0.925, s_1^2 = \frac{\Sigma(x_{1i}-\bar{x}_1)^2}{n_1-1} = .087.$$

Lot 2 : Sample size $n_2 = 10$,

$$\bar{x}_2 = 0.857, s_2^2 = \frac{\Sigma(x_{2i}-\bar{x}_2)^2}{n_2-1} = .079.$$

Do the lots differ significantly in respect of the average impact strength?

Assuming that the lots are of equal variability,

$$t = (\bar{x}_1 - \bar{x}_2) \div \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} = 0.499$$

The 5% value of $|t|$ with $n_1+n_2-2 = 16$ d.f. being 2.120, the data do not lead to rejection of the hypothesis that the two lots have the same average impact strength.

(iii) *Regression problem*

Example: The thickness of zinc coating on 12 pieces of galvanized sheets were determined by the standard stripping method (X) and a magnetic method (Y). The least squares line of regression of Y on X and other statistics were as follows

$$Y = -0.23+1.17x.$$

$S_{xx} = \Sigma x_i^2 - n\bar{x}^2 = 298{,}015$, $S_{yy} = \Sigma y_i^2 - n\bar{y}^2 = 410{,}345$, $S_{xy} = \Sigma x_i y_i - n\bar{x}\bar{y} = 348{,}915$, $b = S_{xy}/S_{xx} = 1.17$, $R_0^2 =$ Residual sum of squares $= S_{yy} - S_{xy}^2/S_{xx} = 1{,}836$. Test if the regression coefficient is significantly higher than 1 at the 1% level.

$$t = (b-1) \div \sqrt{\frac{R_0^2}{(n-2)S_{xx}}} = (1.17-1) \div \sqrt{\frac{1836}{10\times 298015}} = 6.849.$$

The upper 1% value of t with $n-2 = 10$ d.f. being 2.764, the observed regression coefficient is seen to be significantly higher than 1 at the 1% level.

(iv) *Significance of the correlation coefficient*

Example: Is a correlation of $r = 0.52$ between green weight and yield of jute fibre, observed on 20 jute plants significant ?

$$t = \sqrt{n-2} \quad \frac{r}{\sqrt{1-r^2}} = 2.583.$$

The 5% and 1% values of $|t|$ (for two-sided test) with $n-2 = 18$ d.f. being 2.101 and 2.878 respectively, the observed correlation is significant at the 5% level but not at the 1% level. (This test is however valid only under the assumption that the joint distribution of the two variables under study is bivariate normal).

5. Some other tables

1. PEARSON, E. S. and HARTLEY, H. O. (EDS.) (1957): *Biometrika Tables for Statisticians*, Biometrika Trust, Cambridge University Press.

   Table 9 gives the incomplete probability integral of t for $\nu = 1(1)24,30,40,60,120, \infty$; $t = 0(0.1)$ 4(0.2) 8 for $\nu \leqslant 19$ and $= 0(0.05)$ 2 (0.1) 4,5 for $\nu \geqslant 20$.

2. FEDERIGHI, E. T. (1959): Extended Tables of the Percentage Points of Student's t-distribution. *Jour. Amer. Stat. Assn.*, Vol. 54, pp. 683–688.

   Gives t_p to three 3 places of decimal for the following values p and ν.

   $p =$ 0.75, 0.90, 0.95, 0.975, 0.99, 0.995, 0.9975, 0.999, 0.9995, 0.99975, 0.99995, 0.999975, 0.99999, 0.999995, 0.9999975, 0.999999, 0.9999995, 0.99999975, 0.9999999.

   $\nu =$ 1(1)30(5)60(10) 100, 200, 500, 1000, 2000 and 10000.

TABLE 4.1 THE t-DISTRIBUTION: FRACTILES AND CRITICAL VALUES FOR TESTS

ν \ p	0.60	0.70	0.75	0.80	0.85	0.90	0.95	0.975	0.99	0.995	0.999	0.9995
1	.325	.727	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	318.309	636.619
2	.289	.617	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	.277	.584	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	.271	.569	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	.267	.559	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	.265	.553	.718	.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	.263	.549	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	.262	.546	.706	.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	.261	.543	.703	.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	.260	.542	.700	.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	.260	.540	.697	.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	.259	.539	.695	.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	.259	.538	.694	.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	.258	.537	.692	.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	.258	.536	.691	.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	.258	.535	.690	.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	.257	.534	.689	.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	.257	.534	.688	.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	.257	.533	.688	.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	.257	.533	.687	.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	.257	.532	.686	.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	.256	.532	.686	.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	.256	.532	.685	.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	.256	.531	.685	.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	.256	.531	.684	.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	.256	.531	.684	.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	.256	.531	.684	.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	.256	.530	.683	.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	.256	.530	.683	.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	.256	.530	.683	.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	.255	.529	.681	.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	.254	.527	.679	.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	.254	.527	.678	.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	.254	.526	.677	.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
∞	.253	.524	.674	.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
2 sided test	80%	60%	50%	40%	30%	20%	10%	5%	2%	1%	0.2%	0.1%
1 sided test	40%	30%	25%	20%	15%	10%	5%	2.5%	1%	0.5%	0.1%	0.05%
	levels of significance for testing of hypotheses											

Note: 1. ν represents the degrees of freedom.

2. For any given p in the top row, the table provides the value of t_p such that the probability of t being less than t_p is equal to p. For $p < 0.5$, $t_p = -t_{(1-p)}$, $t_{0.50}$ being zero always.

3. For tests of significance, use the critical values for different levels of significance indicated in the last two rows. For a two sided test (for significance of $|t|$) use the levels in the first row. For one sided (upper) test use the levels in the second row. For lower one sided test the critical value is the same as that for the upper tail with the sign changed.

5. THE χ^2-DISTRIBUTION

a. Introduction

Table 5.1 essentially provides, fractiles of the χ^2-distribution for degrees of freedom $\nu = 1\,(1)\,30\,(5)\,40\,(10)\,100$, and for values

$p = 0.005, 0.01, 0.025, 0.05, 0.25, 0.50, 0.75, 0.95, 0.975, 0.99, 0.995.$

Columns (1) and (2) of Table 5.1 gives the lower 1% and 5% values and columns (3) and (4) the upper 1% and 5% values of the distribution of χ^2. These entries are useful in one sided tests using only the upper or the lower tail.

For a two sided test one may use equal partition of tails at any given level of significance. Columns (5) and (6) provide the acceptable interval of χ^2 at 1% level and, columns (7) and (8) that at 5% level. Values of χ^2 beyond the interval on either side will be declared significant.

Columns (9) to (12) provide an alternative set of partitions of χ^2 at the 1% and 5% levels of significance for two sided tests. These are called unbiased partitions (χ_1^2, χ_2^2) and satisfy the equations

$$e^{-\chi_1^2/2}\chi_1^{\nu} = e^{-\chi_2^2/2}\chi_2^{\nu}$$

$$\frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}\int_{\chi_1^2}^{\chi_2^2} e^{-\chi^2/2}\,(\chi^2)^{\frac{\nu-2}{2}}\,d\chi^2 = 1-\alpha = (0.99 \text{ or } 0.95)$$

where ν is the d.f.

The last three columns of Table 5.1 give the first quartile, median and the third quartile of the distribution.

b. Computation of fractiles for other degrees of freedom

The following expansion due to Cornish and Fisher may be used for higher values of ν. χ_p^2 and x are the p-th fractiles of χ^2 (with ν d.f.) and the standard normal distribution respectively. Then.

$$\chi_p^2 = \nu + \sqrt{\nu}\,(x\sqrt{2}) + \frac{2}{3}(x^2-1) + \frac{1}{\sqrt{\nu}}\left(\frac{x^3-7x}{9\sqrt{2}}\right)$$

$$-\frac{1}{\nu}\left(\frac{6x^4+14x^2-32}{405}\right) + \frac{1}{\nu\sqrt{\nu}}\left(\frac{9x^5+256x^3-433x}{4860\sqrt{2}}\right)$$

$$+\frac{1}{\nu^2}\left(\frac{12x^6-243x^4-923x^2+1472}{25515}\right)$$

$$-\frac{1}{\nu^2\sqrt{\nu}}\left(\frac{3753x^7+4353x^5-289517x^3-289717x}{9185400\sqrt{2}}\right)+\cdots$$

Substituting the value of x, from normal tables, χ^2_p can be computed to the desired degree of approximation. To facilitate the computations, the coefficients of $\sqrt{\nu}$, 1, $1/\sqrt{\nu}$ etc. in the above expansion are given below for $p = 0.5$, 0.75, 0.95, 0.975, 0.99, and 0.995. To compute $\chi^2_{(1-p)}$ we use the same tabulated coefficients as for p but with signs of the first, third and every alternate coefficients changed. Thus one can compute χ^2_p for also $p = 0.005$, 0.01, 0.025, 0.05 and 0.25 using the tabulated values of the coefficients.

COEFFICIENTS* IN THE CORNISH-FISHER EXPANSION

coefficient of	value of p					
	0.99	0.95	0.995	0.975	0.5	0.75
$\sqrt{\nu}$	3.2899527	2.3261743	3.6427727	2.7718076	0	0.9538726
1	2.941263	1.137029	3.756598	1.894306	−0.666667	−0.363376
$1/\sqrt{\nu}$	−0.290266	−0.554981	−0.073888	−0.486382	0	−0.346842
$1/\nu$	−0.54197	−0.12296	−0.80252	−0.27240	0.07901	0.06022
$1/\nu\sqrt{\nu}$	0.4116	0.0779	0.6228	0.1948	0	−0.0309
$1/\nu^2$	−0.3425	−0.1006	−0.4642	−0.1952	0.0577	0.0393
$1/\nu^2\sqrt{\nu}$	0.203	0.122	0.183	0.170	0	0.012

* Sufficient figures are retained to ensure accuracy upto the fourth decimal place for $30 < \nu \leq 1600$. For values of $\nu > 1600$, the figures in the first row have to be computed to a higher number of decimal places.

c. Application

Some examples illustrating the use of Table 5.1 are given below.

(i) *Variance of a normal population —tests and confidence intervals*

Example. The sample variance of the blowing time of 10 fuses is :

$$s^2 = \Sigma(x_i - \bar{x})^2/(n-1) = 384.16 \text{ (sec.)}^2.$$

Is this compatible with the hypothesis that the population variance is $\sigma_0^2 = 300$ (sec)^2.

Situation 1 : Given that the population variance can only equal or exceed 300.

$$\frac{(n-1)s^2}{\sigma_0^2} = \frac{9(384.16)}{300} = 11.5248.$$

From Table 5.1 the upper 5% point of χ^2 with $n-1$ (= 9) d.f. is 16.92. Thus the hypothesis cannot be rejected.

Situation 2 : Direction in which deviation from the hypothetical value can occur is unspecified.

If one chooses to apply an unbiased test, the critical values are 2.95 and 20.31. The computed value of χ^2 is well within this interval. Hence the hypothesis cannot be rejected.

On the basis of the observed value of s^2, one can make 95% confidence statements of the following kind.

(a) σ^2 does not exceed $(n-1)s^2/3.33 = 1038.72$

(b) σ^2 is not less than $(n-1)s^2/16.92 = 204.34$

(c) σ^2 lies between $(n-1)s^2/20.31 = 170.23$ and $(n-1)s^2/2.95 = 1172.01$

(d) σ^2 lies between $(n-1)s^2/19.02 = 181.78$ and $(n-1)s^2/2.70 = 1280.53$,

where 3.33 and 16.92 are respectively the lower and upper 5% points, and (2.95, 20.31) and (2.70, 19.02) are respectively the unbiased and equal tail 5% partitions of χ^2, with 9 d.f.

(ii) *Combination of probabilities :* To judge the overall significance of several tests.

Example. The following significance levels were attained in 5 independent tests of the same hypothesis : 0.06, 0.06, 0.07, 0.10, 0.09. Considered together, is the evidence strong enough to reject the hypothesis ?

The appropriate statistic is

$$P_\lambda = -2 \log_e 10 \sum_{i=1}^{k} \log_{10} p_i = 25.993.$$

which, as a χ^2 with $2k\,(= 10)$ d.f., is significant at the 1% level. Hence, even though individually none of the 5 tests leads to rejection of the hypothesis, with the evidence provided by the five independent tests together, the hypothesis stands rejected.

(iii) *Goodness of fit*

For other applications of the χ^2 table in test of goodness of fit, test of independence in contingency tables etc., see some standard books on statistical methods.

d. Some other tables

1. HALD, A. and SINKBAEK. S. A. (1950) : A table of percentage point χ^2 distribution. *Skand Aktuarietidskr,* vol. 33, pp. 168–175.
   Gives fractiles to three places of decimal for the following values of p : 0.0005, 0.001, 0.005, 0.01, 0.025, 0.05, 0.1(0.1) 0.9, 0.95, 0.975, 0.99, 0.995, 0.999, 0.9995 and $\nu =$ 1(1)100.
2. HALD, A. (1952) : *Statistical Tables and Formulas,* John Wiley & Sons, New York.
   Table V gives fractiles to three figures. Otherwise the coverage is same as in 1. above. Table VI gives fractiles of χ^2/ν correct to four places of decimal for the following values of p : 0.0005, 0.001, 0.005, 0.01, 0.025, 0.05, 0.95, 0.975, 0.99, 0.995, 0.999, 0.9995 and $\nu =$ 1(1) 100(5) 200, (10) 300 (50) 1000 (1000) 5000, 10000.
3. PEARSON, E. S. and HARTLEY, H. O. (EDS.) (1957) : *Biometrika Tables for Statisticians,* Biometrica Trust, Cambridge University Press.
   Table 7 gives $\int_{\chi^2}^{\infty} \frac{1}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} e^{-v/2}\, v^{\nu/2-1}\, dv$ to 5 decimal places for $\nu =$ 1(1) 30(2) 70,
   $\chi^2 =$ 0.001 (0.001) 0.01 (0.01) 0.1 (0.1) 2(0.2) 10(0.5) 20(1) 40(2) 134.
   Table 8 gives the fractiles of χ^2 to three and more places of decimal for the following values of p : 0.005, 0.010, 0.025, 0.050, 0.1, 0.25, 0.5, 0.75, 0.9, 0.95, 0.975, 0.995, 0.999 and $\nu =$ 1(1) 30(10) 100.

TABLE 5.1. THE χ^2 DISTRIBUTION: CRITICAL VALUES FOR ONE AND TWO SIDED TESTS AND QUARTILES

df	one sided test				two sided test								quartiles		
	lower tail		upper tail		unbiased partition				partition with equal tail area						
ν	1%	5%	1%	5%	5%		1%		5%		1%		25%	50%	75%
1	$0.0^{3}16$	$0.0^{2}39$	6.63	3.84	$0.0^{2}32$	7.82	$0.0^{3}13$	11.35	$0.0^{3}98$	5.02	$0.0^{4}39$	7.88	0.102	0.455	1.32
2	0.02	0.10	9.21	5.99	0.08	9.53	0.02	13.29	0.05	7.38	0.01	10.60	0.58	1.39	2.77
3	0.11	0.35	11.34	7.81	0.30	11.19	0.10	15.13	0.22	9.35	0.07	12.84	1.21	2.37	4.11
4	0.30	0.71	13.28	9.49	0.61	12.80	0.26	16.90	0.48	11.14	0.21	14.86	1.92	3.36	5.39
5	0.55	1.15	15.09	11.07	0.99	14.37	0.50	18.63	0.83	12.83	0.41	16.75	2.67	4.35	6.63
6	0.87	1.64	16.81	12.59	1.43	15.90	0.79	20.30	1.24	14.45	0.68	18.55	3.45	5.35	7.84
7	1.24	2.17	18.48	14.07	1.90	17.39	1.12	21.93	1.69	16.01	0.99	20.28	4.25	6.35	9.04
8	1.65	2.73	20.09	15.51	2.41	18.86	1.50	23.53	2.18	17.53	1.34	21.96	5.07	7.34	10.22
9	2.09	3.33	21.67	16.92	2.95	20.31	1.91	25.11	2.70	19.02	1.73	23.59	5.90	8.34	11.39
10	2.56	3.94	23.21	18.31	3.52	21.73	2.34	26.65	3.25	20.48	2.16	25.19	6.74	9.34	12.55
11	3.05	4.57	24.72	19.68	4.10	23.13	2.81	28.18	3.82	21.92	2.60	26.76	7.58	10.34	13.70
12	3.57	5.23	26.22	21.03	4.70	24.52	3.29	29.68	4.40	23.34	3.07	28.30	8.44	11.34	14.85
13	4.11	5.89	27.69	22.36	5.32	25.90	3.79	31.17	5.01	24.74	3.57	29.82	9.30	12.34	15.98
14	4.66	6.57	29.14	23.68	5.95	27.26	4.32	32.64	5.63	26.12	4.07	31.32	10.17	13.34	17.12
15	5.23	7.26	30.58	25.00	6.59	28.61	4.85	34.10	6.26	27.49	4.60	32.80	11.04	14.34	18.25
16	5.81	7.96	32.00	26.30	7.25	29.96	5.40	35.54	6.91	28.85	5.14	34.27	11.91	15.34	19.37
17	6.41	8.67	33.41	27.59	7.91	31.29	5.97	36.97	7.56	30.19	5.70	35.72	12.79	16.34	20.49
18	7.01	9.39	34.81	28.87	8.58	32.61	6.54	38.39	8.23	31.53	6.26	37.16	13.68	17.34	21.60
19	7.63	10.12	36.19	30.14	9.27	33.92	7.13	39.80	8.91	32.85	6.84	38.58	14.56	18.34	22.72
20	8.26	10.85	37.57	31.41	9.96	35.23	7.73	41.20	9.59	34.17	7.43	40.00	15.45	19.34	23.83
21	8.90	11.59	38.93	32.67	10.66	36.52	8.34	42.59	10.28	35.48	8.03	41.40	16.34	20.34	24.93
22	9.54	12.34	40.29	33.92	11.36	37.82	8.95	43.97	10.98	36.78	8.64	42.80	17.24	21.34	26.04
23	10.20	13.09	41.64	35.17	12.07	39.10	9.58	45.34	11.69	38.08	9.26	44.18	18.14	22.34	27.14
24	10.86	13.85	42.98	36.42	12.79	40.38	10.21	46.71	12.40	39.36	9.89	45.56	19.04	23.34	28.24
25	11.52	14.61	44.31	37.65	13.51	41.66	10.85	48.06	13.12	40.65	10.52	46.93	19.94	24.34	29.34
26	12.20	15.38	45.64	38.89	14.24	42.93	11.49	49.42	13.84	41.92	11.16	48.29	20.84	25.34	30.43
27	12.88	16.15	46.96	40.11	14.98	44.19	12.14	50.76	14.57	43.19	11.81	49.64	21.75	26.34	31.53
28	13.56	16.93	48.28	41.34	15.72	45.45	12.80	52.10	15.31	44.46	12.46	50.99	22.66	27.34	32.62
29	14.26	17.71	49.59	42.56	16.46	46.71	13.47	53.43	16.05	45.72	13.12	52.34	23.57	28.34	33.71
30	14.95	18.49	50.89	43.77	17.21	47.96	14.14	54.76	16.79	46.98	13.79	53.67	24.48	29.34	34.80
35	18.51	22.47	57.34	49.80	21.00	54.16	17.56	61.33	20.57	53.20	17.19	60.27	29.05	34.34	40.22
40	22.16	26.51	63.69	55.76	24.88	60.27	21.09	67.79	24.43	59.34	20.71	66.77	33.66	39.34	45.62
50	29.71	34.77	76.15	67.51	32.82	72.32	28.40	80.47	32.36	71.42	27.99	79.49	42.94	49.33	56.33
60	37.48	43.19	88.38	79.08	40.97	84.18	35.97	92.91	40.48	83.30	35.53	91.95	52.29	59.33	66.98
70	45.44	51.74	100.42	90.53	49.25	95.89	43.72	105.15	48.76	95.02	43.28	104.22	61.70	69.33	77.58
80	53.54	60.39	112.33	101.88	57.66	107.48	51.63	117.23	57.15	106.63	51.17	116.32	71.14	79.33	88.13
90	61.75	69.13	124.12	113.15	66.16	118.98	59.67	129.20	65.65	118.14	59.20	128.30	80.62	89.33	98.65
100	70.06	77.93	135.81	124.34	74.74	130.39	67.81	141.05	74.22	129.56	67.33	140.17	90.13	99.33	109.14

Note: For significance χ^2 should exceed tabulated value for one sided upper tail test, χ^2 should be less than tabulated value for one sided lower tail test and χ^2 should be outside tabulated interval for a two sided test.

6. THE F DISTRIBUTION

6.1. Fractiles

a. Introduction

Table 6.1 gives fractiles of the F distribution for various combinations of ν_1 and ν_2, the degrees of freedom of the numerator and denominator mean squares respectively. The values of p and the degrees of freedom covered are :

$$p = 0.25 \quad 0.5,\ 0.75,\ 0.95,\ 0.975,\ 0.99,\ 0.995$$

$$\nu_1 = 1(1)9,\ 12,\ 24,\ \infty$$

$$\nu_2 = 1(1)30,\ 40,\ 60,\ 120,\ \infty.$$

If F_p (ν_1, ν_2) denotes the p-th fractile, then we have the relation F_{1-p} (ν_1, ν_2) $= 1/F_p(\nu_2, \nu_1)$, so that Table 6.I can be used to obtain the fractiles for $p = .005$, 0.01 0.025, 0.05 (i.e. the lower 0.5%, 1%, 2.5% and 5% points of F) as shown in example below.

Example. To find $F_p(\nu_1, \nu_2)$ for $\nu_1 = 4$, $\nu_2 = 8$, $p = 0.05$.

The required fractile is $1/6.04 = 0.166$, the value 6.04 being the upper 5% point of F with $\nu_1 = 8$ and $\nu_2 = 4$ d.f.

b. Interpolation in Table 6.1 (ν_1–and ν_2–wise)

In Table 6.1, the larger values of ν_1 and ν_2 have been chosen to be in harmonic progression. This is because, for large values of ν_1 and ν_2, quadratic or even linear interpolation, with the reciprocal of the d.f. as the argument, is sufficiently accurate.

Formulae for harmonic interpolation

ν_1—wise	linear	ν_1—wise	quadratic
$9 < \nu_1 < 12$	$(1-u^*)y_8+u^*y_{12}$	$10 < \nu_1 \leqslant 16$	$\frac{u(u+1)}{2}y_8-(u^2-1)y_{12}+\frac{u(u-1)}{2}y_{24}$
$12 < \nu_1 < 24$	$(1-u^*)y_{12}+u^*y_{24}$	$\nu_1 \geqslant 17$	$\frac{u(u+1)}{2}y_{12}-(u^2-1)y_{24}+\frac{u(u-1)}{2}y_{\infty}$
$\nu_1 > 24$	$(1-u^*)y_{24}+u^*y_{\infty}$		

ν_2—wise	linear	ν_2—wise	quadratic
$30 < \nu_2 < 40$	$(1-u^*)y_{30}+u^*y_{40}$	$31 \leqslant \nu_2 \leqslant 34$	$\frac{u(u+1)}{2}y_{24}-(u^2-1)y_{30}+\frac{u(u-1)}{2}y_{40}$
$40 < \nu_2 < 60$	$(1-u^*)y_{40}+u^*y_{60}$	$35 \leqslant \nu_2 \leqslant 48$	$\frac{u(u+1)}{2}y_{30}-(u^2-1)y_{40}+\frac{u(u-1)}{2}y_{60}$
$60 < \nu_2 < 120$	$(1-u^*)y_{60}+u^*y_{120}$	$49 \leqslant \nu_2 \leqslant 80$	$\frac{u(u+1)}{2}y_{40}-(u^2-1)y_{60}+\frac{u(u-1)}{2}y_{120}$
		$81 \leqslant \nu_2 \leqslant 119$	$\frac{u(u+1)}{2}y_{60}-(u^2-1)y_{120}+\frac{u(u-1)}{2}y_{\infty}$

Note : (1) $u^*=u$ if $u \geqslant 0, =1+u$ if $u < 0$

(e) y_k is the tabulated value for $\nu_1=k$ in the formulae for ν_1—wise interpolation and for $\nu_2 = k$ in the formulae for ν_2 wise interpolation

VALUES OF u FOR INTERPOLATION IN TABLE 6.1

1. $\nu_1 = 8(1)60$

ν_1	u	ν_1	u	ν_1	u	ν_1	u	ν_1	u	ν_1	u
8	0	18	0.3333	28	−0.1429	38	−0.3684	48	−0.5000	58	0.4138
9		19	0.2632	29	−0.1724	39	−0.3846	49	0.4898	59	0.4068
10	0.4000	20	0.2000	30	−0.2000	40	−0.4000	50	0.4800	60	0.4000
11	0.1818	21	0.1429	31	−0.2258	41	−0.4146	51	0.4706		
12	0	22	0.0909	32	−0.2500	42	−0.4286	52	0.4615		
13	−0.1538	23	0.0435	33	−0.2727	43	−0.4419	53	0.4528		
14	−0.2857	24	0	34	−0.2941	44	−0.4546	54	0.4444		
15	−0.4000	25	−0.0400	35	−0.3143	45	−0.4667	55	0.4364		
16	−0.5000	26	−0.0769	36	−0.3333	46	−0.4783	56	0.4286		
17	0.4118	27	−0.1111	37	−0.3514	47	−0.4894	57	0.4210		

2. $\nu_2 = 30(1)120$

ν_2	u	ν_2	u	ν_2	u	ν_2	u	ν_2	u	ν_2	u
30	0	45	−0.3333	60	0	75	−0.4000	90	0.3333	105	0.1429
31	−0.1290	46	−0.3913	61	−0.0328	76	−0.4211	91	0.3187	106	0.1321
32	−0.2500	47	−0.4468	62	−0.0645	77	−0.4416	92	0.3043	107	0.1215
33	−0.3636	48	−0.5000	63	−0.0952	78	−0.4615	93	0.2903	108	0.1111
34	−0.4706	49	0.4490	64	−0.1250	79	−0.4810	94	0.2766	109	0.1009
35	0.4286	50	0.4000	65	−0.1539	80	−0.5000	95	0.2632	110	0.0909
36	0.3333	51	0.3529	66	−0.1818	81	0.4815	96	0.2500	111	0.0811
37	0.2432	52	0.3077	67	−0.2090	82	0.4634	97	0.2371	112	0.0714
38	0.1579	53	0.2641	68	−0.2353	83	0.4458	98	0.2245	113	0.0619
39	0.0769	54	0.2222	69	−0.2609	84	0.4286	99	0.2121	114	0.0526
40	0	55	0.1818	70	−0.2857	85	0.4118	100	0.2000	115	0.0435
41	−0.0732	56	0.1429	71	−0.3099	86	0.3953	101	0.1881	116	0.0345
42	−0.1429	57	0.1053	72	−0.3333	87	0.3793	102	0.1765	117	0.0256
43	−0.2092	58	0.0690	73	−0.3562	88	0.3636	103	0.1650	118	0.0169
44	−0.2727	59	0.0339	74	−0.3784	89	0.3483	104	0.1538	119	0.0084

Example. To compute $F_p(\nu_1, \nu_2)$ for $\nu_1 = 6$, $\nu_2 = 44$, $p = 0.95$.

A ν_2-wise interpolation is necessary. For $\nu_2 = 44$, we have $u = -0.2727$, and $u^* = 1+u = .7273$. Also from Table 6.1 we have $y_{40} = 2.34$ and $y_{60} = 2.25$. Hence the required value

$$y_{44} = (1+u^*)y_{40} + u^* y_{60} = 2.315.$$

For higher accuracy the Cornish-Fisher expansion of z_p (the p-th fractile of $z = \frac{1}{2}\log_e F$) may be used.

$$z_p = x\sqrt{\left(\frac{\sigma}{2}\right)} - \delta\left(\frac{x^2+2}{6}\right) + \sqrt{\left(\frac{\sigma}{2}\right)}\left\{\sigma\left(\frac{x^3+3x}{24}\right) + \frac{\delta^2}{\sigma}\left(\frac{x^3+11x}{72}\right)\right\}$$

$$-\left\{\delta\sigma\left(\frac{x^4+9x^2+8}{120}\right) - \frac{\delta^3}{\sigma}\left(\frac{3x^4+7x^2-16}{3240}\right)\right\} + \sqrt{\left(\frac{\sigma}{2}\right)}\left\{\sigma^2\left(\frac{x^5+20x^3+15x}{1920}\right.\right.$$

$$\left.\left. + \delta^2\left(\frac{x^5+44x^3+183x}{2880}\right) + \frac{\delta^4}{\sigma^2}\left(\frac{9x^5-284x^3-1513x}{155520}\right)\right\}\right.$$

$$+\left\{\delta\sigma^2\left(\frac{4x^6-25x^4-177x^2+192}{20160}\right) + \delta^3\left(\frac{4x^6+101x^4+117x^2-480}{90720}\right)\right.$$

$$\left. - \frac{\delta^5}{\sigma^2}\left(\frac{12x^6+513x^4+841x^2-2560}{1632960}\right)\right\} + \ldots \ldots \ldots$$

where x is the p-th fractile of the standard normal distribution,

$$\sigma = \frac{1}{\nu_1} + \frac{1}{\nu_2}, \quad \delta = \frac{1}{\nu_1} - \frac{1}{\nu_2}$$

The coefficients in the expansion are given below for selected values of p.

COEFFICIENTS IN THE CORNISH-FISHER EXPANSION

coefficient of	value of p					
	0.5	0.75	0.95	0.975	0.99	0.995
$\sqrt{\sigma/2}$	0	0.67448975	1.64485363	1.95996398	2.32634787	2.57582930
$-\delta$	0.33333333	0.40915607	0.78425724	0.97357647	1.23531574	1.43914943
$\sigma\sqrt{\sigma/2}$	0	0.0970966	0.3910327	0.5587089	0.8153747	1.0340770
$\delta^2/\sqrt{2\sigma}$	0	0.1073089	0.3131057	0.4040101	0.5302747	0.6308956
$-\delta\sigma$	0.0666667	0.1025116	0.3305821	0.4777495	0.7166304	0.9311327
δ^3/σ	-0.004938	-0.003764	0.007685	0.017025	0.033873	0.050157
$\sigma^2\sqrt{\sigma/2}$	0	0.008539	0.065478	0.108805	0.184807	0.257207
$\delta^2\sqrt{\sigma/2}$	0	0.047595	0.176687	0.249610	0.363825	0.464148
$\delta^4/\sqrt{2\sigma^3}$	0	-0.00711	-0.02343	-0.03114	-0.04168	-0.04971
$\delta\sigma^2$	0.00952	0.00529	-0.01938	-0.03126	-0.04286	-0.04537
δ^3	-0.00529	-0.00447	0.00722	0.01859	0.04128	0.06515
$-\delta^5/\sigma^2$	0	0	0	0	0	0.0178
$-\sigma^3\sqrt{\sigma/2}$	0	0.00344	0.01491	0.02660	0.5478	0.09004
$\delta\sigma^2$	0	0.0109	0.0804	0.1534	0.3174	0.5105

Sufficient digits have been retained so as to ensure accuracy in the sixth place of decimal for $\nu_1 > 24$ and $\nu_2 > 60$.

c. Applications

Some uses of Table 6.1 are illustrated in the following examples.

(i) *Ratio of Variances—tests and confidence intervals*

Example. Use the data given in subsection **c** of chapter 4 to test if the two lots reveal equal variability in respect of impact strength. Denoting the variances of impact strength in lots 1 and 2 by σ_1^2 and σ_2^2 respectively, the problem reduces to testing $\theta = \sigma_1^2/\sigma_2^2 = 1$. To test against alternatives $\sigma_1^2 \neq \sigma_2^2$ compute F by putting the larger mean square in the numerator and compare it with the upper 2.5% value of F with the corresponding degrees of freedom. Thus $F = .087/.079 = 1.101$. The upper 2.5% value of F (with $\nu_1 = 7$ and $\nu_2 = 9$) is 4.20. Hence the hypothesis $\theta = 1$ cannot be rejected on the basis of the given data.

One can make 95% confidence statements of the following kind.

(a) σ_1^2/σ_2^2 does not exceed $s_1^2/s_2^2 \div 0.27 = 4.08$

(b) σ_1^2/σ_2^2 is not less than $s_1^2/s_2^2 \div 3.29 = 0.33$

(c) σ_1^2/σ_2^2 lies between $s_1^2/s_2^2 \div 4.20 = 0.26$ and $s_1^2/s_2^2 \div 0.21 = 5.24$.

where 0.27 and 3.29 are respectively the lower and upper 5% points, and 0.21 and 4.20 the lower and upper 2.5% points of F with $\nu_1 = 7$ and $\nu_2 = 9$.

(ii) *Analysis of variance—one-way classification*

Example. Five sets of six mixes, each mix providing 24 doughnuts, were cooked in five types of fats. The table below gives in grams the fat absorbed per mix. Test if the amount of fat absorbed is a characteristic of the type of fat used for cooking.

GRAMS OF FAT ABSORBED BY MIX OF 24 DOUGHNUTS

	type of fat				
	1	2	3	4	5
	24	33	37	38	23
	32	21	43	51	25
	28	50	57	57	4
	37	40	29	42	37
	16	57	39	45	25
	55	27	47	37	36
total	192	228	252	270	150

Grand total $G = 1092$. Total number of observations, $n = 30$.

Correction facter (C.F.) $= G^2/n = G^2/30 = 39748.8$

Total S.S. $= 24^2+32^2+28^2+\ldots+25^2+36^2-\text{C.F.} = 44592.0-39748.8 = 4843.2$

S.S. due to fats $= \dfrac{T_1^2}{n_1}+\dfrac{T_2^2}{n_2}+\ldots+\dfrac{T_k^2}{n_k}-\text{C.F.}$ (where T_i is the total for the i-th fat with n_i observations)

$$= \frac{1}{6}(192^2+228^2+\ldots+150^2)-\text{C.F.} = 41292.0-39748.8 = 1543.2.$$

ANALYSIS OF VARIANCE TABLE

sources of variation	d.f.	s.s.	m.s.	F = ratio of m.s.
between fats	4	1543.2	385.8	2.922*
within fats	25	3300.0†	132.0	
total	29	4843.2		

† obtained by subtraction.

The upper 5% and 1% values of F (for $\nu_1 = 4$, $\nu_2 = 25$) are 2.76 and 4.18 respectively. The results are thus significant at the 5% level and it may be concluded that the amount of fat absorption depends on the fat used for cooking.

(iii) *Multiple correlation—test of significance*

The multiple correlation coefficient between rate of gain in weight (x_1) and two other variables, initial weight (x_2) and age (x_3), was $R_{1\cdot23} = 0.421$, based on observations on 40 swines.

To test for its significance, compute

$$\frac{n-k-1}{k} \quad \frac{R^2}{1-R^2} = \frac{37}{2} \quad \frac{(0.421)^2}{1-(0.421)^2} = 3.991$$

where k is the number of independent variables, and n is the sample size.

The upper 5% and 1% values of F (with $\nu_1 = k = 2$ and $\nu_2 = n-k-1 = 37$) are 3.25 and 5.23 respectively (values obtained by interpolation). Hence the observed values of $R_{1\cdot23}$ is significant at the 5% level (though not at the 1% level).

(iv) *Test of mean values in multivariate normal populations*

Example. Differences d_1 and d_2 in head length and head breadth between first-born and second-born sons were observed on 25 families. Test if the first-born in a family differs significantly from the second-born, in respect of these two characteristics.

The following values were obtained from the data

Mean difference : $\bar{d}_1 = 1.88$, $\bar{d}_2 = 1.48$.

The dispersion matrix of the differences estimated on 24 d.f. (obtained by dividing the corrected sum of squares and products by 24) is given by

$$w_{11} = 68.03, \; w_{12} = 11.52, \; w_{22} = 24.01$$

The inverse of this matrix is,

$$w^{11} = 0.0159999, \quad w^{12} = -0.007677, \quad w^{22} = 0.045332.$$

The problem is equivalent to testing if the sample mean vector ($\bar{d}_1$, $\bar{d}_2$) differs significantly from (0, 0). The appropriate statistic (which is distributed as F on k and $n-k$ d.f.) is

$$\frac{n-k}{(n-1)k} \left[n\Sigma \, \Sigma w^{ij} \, \bar{d}_i \, \bar{d}_j \right] = \frac{23}{2} \cdot \frac{25}{24} (0.113121) = 1.3548.$$

where n is the sample size and k is the number of variables. Note that $n(w^{ij})$ is the inverse of the estimated dispersion matrix of $\bar{d}_1$ and $\bar{d}_2$. The upper 5% value of F (with $\nu_1 = k = 2$ and $\nu_2 = n-k = 23$) $= 3.42$. Since 1.3548 is less than this value, it is concluded that the data do not provide evidence of differences in the dimensions of the firstborn and second-born sons.

d. Another table

1. Merrington, M. and Thompson, C. M. (1943) : Tables of percentage points of the inverted beta (F) distribution, *Biometrika*, **33**, 73–88.

   Gives to 5 figures fractiles of the F distribution for the following values of p, ν_1, and ν_2.

   $p = 0.50, 0.75, 0.90, 0.95, 0.975, 0.99, 0.995.$

   $\nu_1 = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$

   $\nu_2 = 1(1)30, 40, 60, 120, \infty.$

TABLE 6.1. THE F DISTRIBUTION : FRACTILES

	$\nu_1 = 1$								$\nu_1 = 2$								
ν_2	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	ν_2
1	0.17	1.00	5.83	39.86	161.4	647.8	4052	16211	0.39	1.50	7.50	49.50	199.5	799.5	4999.5	20000	1
2	0.13	0.67	2.57	8.53	18.51	38.51	98.50	198.5	0.33	1.00	3.00	9.00	19.00	39.00	99.00	199.0	2
3	0.12	0.59	2.02	5.54	10.13	17.44	34.12	55.55	0.32	0.88	2.28	5.46	9.55	16.04	30.82	49.80	3
4	0.12	0.55	1.81	4.54	7.71	12.22	21.20	31.33	0.31	0.83	2.00	4.32	6.94	10.65	18.00	26.28	4
5	0.11	0.53	1.69	4.06	6.61	10.01	16.26	22.78	0.30	0.80	1.85	3.78	5.79	8.43	13.27	18.31	5
6	0.11	0.51	1.62	3.78	5.99	8.81	13.75	18.63	0.30	0.78	1.76	3.46	5.14	7.26	10.92	14.54	6
7	0.11	0.51	1.57	3.59	5.59	8.07	12.25	16.24	0.30	0.77	1.70	3.26	4.74	6.54	9.55	12.40	7
8	0.11	0.50	1.54	3.46	5.32	7.57	11.26	14.69	0.30	0.76	1.66	3.11	4.46	6.06	8.65	11.04	8
9	0.11	0.49	1.51	3.36	5.12	7.21	10.56	13.61	0.30	0.75	1.62	3.01	4.26	5.71	8.02	10.11	9
10	0.11	0.49	1.49	3.29	4.96	6.94	10.04	12.83	0.30	0.74	1.60	2.92	4.10	5.46	7.56	9.43	10
11	0.11	0.49	1.47	3.23	4.84	6.72	9.65	12.23	0.30	0.74	1.58	2.86	3.98	5.26	7.21	8.91	11
12	0.11	0.48	1.46	3.18	4.75	6.55	9.33	11.75	0.29	0.73	1.56	2.81	3.89	5.10	6.93	8.51	12
13	0.11	0.48	1.45	3.14	4.67	6.41	9.07	11.37	0.29	0.73	1.55	2.76	3.81	4.97	6.70	8.19	13
14	0.11	0.48	1.44	3.10	4.60	6.30	8.86	11.06	0.29	0.73	1.53	2.73	3.74	4.86	6.51	7.92	14
15	0.11	0.48	1.43	3.07	4.54	6.20	8.68	10.80	0.29	0.73	1.52	2.70	3.68	4.77	6.36	7.70	15
16	0.11	0.48	1.42	3.05	4.49	6.12	8.53	10.58	0.29	0.72	1.51	2.67	3.63	4.69	6.23	7.51	16
17	0.10	0.47	1.42	3.03	4.45	6.04	8.40	10.38	0.29	0.72	1.51	2.64	3.59	4.62	6.11	7.35	17
18	0.10	0.47	1.41	3.01	4.41	5.98	8.29	10.22	0.29	0.72	1.50	2.62	3.55	4.56	6.01	7.21	18
19	0.10	0.47	1.41	2.99	4.38	5.92	8.18	10.07	0.29	0.72	1.49	2.61	3.52	4.51	5.93	7.09	19
20	0.10	0.47	1.40	2.97	4.35	5.87	8.10	9.94	0.29	0.72	1.49	2.59	3.49	4.46	5.85	6.99	20
21	0.10	0.47	1.40	2.96	4.32	5.83	8.02	9.83	0.29	0.72	1.48	2.57	3.47	4.42	5.78	6.89	21
22	0.10	0.47	1.40	2.95	4.30	5.79	7.95	9.73	0.29	0.72	1.48	2.56	3.44	4.38	5.72	6.81	22
23	0.10	0.47	1.39	2.94	4.28	5.75	7.88	9.63	0.29	0.71	1.47	2.55	3.42	4.35	5.66	6.73	23
24	0.10	0.47	1.39	2.93	4.26	5.72	7.82	9.55	0.29	0.71	1.47	2.54	3.40	4.32	5.61	6.66	24
25	0.10	0.47	1.39	2.92	4.24	5.69	7.77	9.48	0.29	0.71	1.47	2.53	3.39	4.29	5.57	6.60	25
26	0.10	0.47	1.38	2.91	4.23	5.66	7.72	9.41	0.29	0.71	1.46	2.52	3.37	4.27	5.53	6.54	26
27	0.10	0.47	1.38	2.90	4.21	5.63	7.68	9.34	0.29	0.71	1.46	2.51	3.35	4.24	5.49	6.49	27
28	0.10	0.47	1.38	2.89	4.20	5.61	7.64	9.28	0.29	0.71	1.46	2.50	3.34	4.22	5.45	6.44	28
29	0.10	0.47	1.38	2.89	4.18	5.59	7.60	9.23	0.29	0.71	1.45	2.50	3.33	4.20	5.42	6.40	29
30	0.10	0.47	1.38	2.88	4.17	5.57	7.56	9.18	0.29	0.71	1.45	2.49	3.32	4.18	5.39	6.35	30
40	0.10	0.46	1.36	2.84	4.08	5.42	7.31	8.83	0.29	0.71	1.44	2.44	3.23	4.05	5.18	6.07	40
60	0.10	0.46	1.35	2.79	4.00	5.29	7.08	8.49	0.29	0.70	1.42	2.39	3.15	3.93	4.98	5.79	60
120	0.10	0.46	1.34	2.75	3.92	5.15	6.85	8.18	0.29	0.70	1.40	2.35	3.07	3.80	4.79	5.54	120
∞	0.10	0.45	1.32	2.71	3.84	5.02	6.63	7.88	0.29	0.69	1.39	2.30	3.00	3.69	4.61	5.30	∞
level of significance				10%	5%	2.5%	1%	0.5%				10%	5%	2.5%	1%	0.5%	
				one sided test (upper tail)								one sided test (upper tail)					

TABLE 6.1. *(continued)*. THE F DISTRIBUTION : FRACTILES

	$\nu_1 = 3$								$\nu_1 = 4$								
ν_2	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	ν_2
1	0.49	1.71	8.20	53.59	215.7	864.2	5403	21615	0.55	1.82	8.58	55.83	224.6	899.6	5625	22500	1
2	0.44	1.13	3.15	9.16	19.16	39.17	99.17	199.2	0.50	1.21	3.23	9.24	19.25	39.25	99.25	199.2	2
3	0.42	1.00	2.36	5.39	9.28	15.44	29.46	47.47	0.49	1.06	2.39	5.34	9.12	15.10	28.71	46.19	3
4	0.42	0.94	2.05	4.19	6.59	9.98	16.69	24.26	0.48	1.00	2.06	4.11	6.39	9.60	15.98	23.15	4
5	0.42	0.91	1.88	3.62	5.41	7.76	12.06	16.53	0.48	0.96	1.89	3.52	5.19	7.39	11.39	15.56	5
6	0.41	0.89	1.78	3.29	4.76	6.60	9.78	12.92	0.48	0.94	1.79	3.18	4.53	6.23	9.15	12.03	6
7	0.41	0.87	1.72	3.07	4.35	5.89	8.45	10.88	0.48	0.93	1.72	2.96	4.12	5.52	7.85	10.05	7
8	0.41	0.86	1.67	2.92	4.07	5.42	7.59	9.60	0.48	0.91	1.66	2.81	3.84	5.05	7.01	8.81	8
9	0.41	0.85	1.63	2.81	3.86	5.08	6.99	8.72	0.48	0.91	1.63	2.69	3.63	4.72	6.42	7.96	9
10	0.41	0.85	1.60	2.73	3.71	4.83	6.55	8.08	0.48	0.90	1.59	2.61	3.48	4.47	5.99	7.34	10
11	0.41	0.84	1.58	2.66	3.59	4.63	6.22	7.60	0.48	0.89	1.57	2.54	3.36	4.28	5.67	6.88	11
12	0.41	0.84	1.56	2.61	3.49	4.47	5.95	7.23	0.48	0.89	1.55	2.48	3.26	4.12	5.41	6.52	12
13	0.41	0.83	1.55	2.56	3.41	4.35	5.74	6.93	0.48	0.88	1.53	2.43	3.18	4.00	5.21	6.23	13
14	0.41	0.83	1.53	2.52	3.34	4.24	5.56	6.68	0.48	0.88	1.52	2.39	3.11	3.89	5.04	6.00	14
15	0.41	0.83	1.52	2.49	3.29	4.15	5.42	6.48	0.48	0.88	1.51	2.36	3.06	3.80	4.89	5.80	15
16	0.41	0.82	1.51	2.46	3.24	4.08	5.29	6.30	0.48	0.88	1.50	2.33	3.01	3.73	4.77	5.64	16
17	0.41	0.82	1.50	2.44	3.20	4.01	5.18	6.16	0.48	0.87	1.49	2.31	2.96	3.66	4.67	5.50	17
18	0.41	0.82	1.49	2.42	3.16	3.95	5.09	6.03	0.48	0.87	1.48	2.29	2.93	3.61	4.58	5.37	18
19	0.41	0.82	1.49	2.40	3.13	3.90	5.01	5.92	0.48	0.87	1.47	2.27	2.90	3.56	4.50	5.27	19
20	0.41	0.82	1.48	2.38	3.10	3.86	4.94	5.82	0.48	0.87	1.47	2.25	2.87	3.51	4.43	5.17	20
21	0.41	0.81	1.48	2.36	3.07	3.82	4.87	5.73	0.48	0.87	1.46	2.23	2.84	3.48	4.37	5.09	21
22	0.41	0.81	1.47	2.35	3.05	3.78	4.82	5.65	0.48	0.87	1.45	2.22	2.82	3.44	4.31	5.02	22
23	0.41	0.81	1.47	2.34	3.03	3.75	4.76	5.58	0.48	0.86	1.45	2.21	2.80	3.41	4.26	4.95	23
24	0.41	0.81	1.46	2.33	3.01	3.72	4.72	5.52	0.48	0.86	1.44	2.19	2.78	3.38	4.22	4.89	24
25	0.41	0.81	1.46	2.32	2.99	3.69	4.68	5.46	0.48	0.86	1.44	2.18	2.76	3.35	4.18	4.84	25
26	0.41	0.81	1.45	2.31	2.98	3.67	4.64	5.41	0.48	0.86	1.44	2.17	2.74	3.33	4.14	4.79	26
27	0.41	0.81	1.45	2.30	2.96	3.65	4.60	5.36	0.48	0.86	1.43	2.17	2.73	3.31	4.11	4.74	27
28	0.41	0.81	1.45	2.29	2.95	3.63	4.57	5.32	0.48	0.86	1.43	2.16	2.71	3.29	4.07	4.70	28
29	0.41	0.81	1.45	2.28	2.93	3.61	4.54	5.28	0.48	0.86	1.43	2.15	2.70	3.27	4.04	4.66	29
30	0.41	0.81	1.44	2.28	2.92	3.59	4.51	5.24	0.48	0.86	1.42	2.14	2.69	3.25	4.02	4.62	30
40	0.41	0.80	1.42	2.23	2.84	3.46	4.31	4.98	0.48	0.85	1.40	2.09	2.61	3.13	3.83	4.37	40
60	0.40	0.80	1.41	2.18	2.76	3.34	4.13	4.73	0.48	0.85	1.38	2.04	2.53	3.01	3.65	4.14	60
120	0.40	0.79	1.39	2.13	2.68	3.23	3.95	4.50	0.48	0.84	1.37	1.99	2.45	2.89	3.48	3.92	120
∞	0.40	0.79	1.37	2.08	2.60	3.12	3.78	4.28	0.48	0.84	1.35	1.94	2.37	2.79	3.32	3.72	∞
level of significance				10%	5%	2.5%	1%	0.5%				10%	5%	2.5%	1%	0.5%	
				one sided test (upper tail)								one sided test (upper tail)					

TABLE 6.1. (*continued*). THE F DISTRIBUTION : FRACTILES

	$\nu_1 = 5$								$\nu_1 = 6$								
ν_2	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	ν_2
1	0.59	1.89	8.82	57.24	230.2	921.8	5764	23056	0.62	1.94	8.98	58.20	234.0	937.1	5859	23437	1
2	0.54	1.25	3.28	9.29	19.30	39.30	99.30	199.3	0.57	1.28	3.31	9.33	19.33	39.33	99.33	199.3	2
3	0.53	1.10	2.41	5.31	9.01	14.88	28.24	45.39	0.56	1.13	2.42	5.28	8.94	14.73	27.91	44.84	3
4	0.53	1.04	2.07	4.05	6.26	9.36	15.52	22.46	0.56	1.06	2.08	4.01	6.16	9.20	15.21	21.97	4
5	0.53	1.00	1.89	3.45	5.05	7.15	10.97	14.94	0.56	1.02	1.89	3.40	4.95	6.98	10.67	14.51	5
6	0.53	0.98	1.79	3.11	4.39	5.99	8.75	11.46	0.56	1.00	1.78	3.05	4.28	5.82	8.47	11.07	6
7	0.53	0.96	1.71	2.88	3.97	5.29	7.46	9.52	0.56	0.98	1.71	2.83	3.87	5.12	7.19	9.16	7
8	0.53	0.95	1.66	2.73	3.69	4.82	6.63	8.30	0.56	0.97	1.65	2.67	3.58	4.65	6.37	7.95	8
9	0.53	0.94	1.62	2.61	3.48	4.48	6.06	7.47	0.56	0.96	1.61	2.55	3.37	4.32	5.80	7.13	9
10	0.53	0.93	1.59	2.52	3.33	4.24	5.64	6.87	0.56	0.95	1.58	2.46	3.22	4.07	5.39	6.54	10
11	0.53	0.93	1.56	2.45	3.20	4.04	5.32	6.42	0.57	0.95	1.55	2.39	3.09	3.88	5.07	6.10	11
12	0.53	0.92	1.54	2.39	3.11	3.89	5.06	6.07	0.57	0.94	1.53	2.33	3.00	3.73	4.82	5.76	12
13	0.53	0.92	1.52	2.35	3.03	3.77	4.86	5.79	0.57	0.94	1.51	2.28	2.92	3.60	4.62	5.48	13
14	0.53	0.91	1.51	2.31	2.96	3.66	4.69	5.56	0.57	0.94	1.50	2.24	2.85	3.50	4.46	5.26	14
15	0.53	0.91	1.49	2.27	2.90	3.58	4.56	5.37	0.57	0.93	1.48	2.21	2.79	3.41	4.32	5.07	15
16	0.53	0.91	1.48	2.24	2.85	3.50	4.44	5.21	0.57	0.93	1.47	2.18	2.74	3.34	4.20	4.91	16
17	0.53	0.91	1.47	2.22	2.81	3.44	4.34	5.07	0.57	0.93	1.46	2.15	2.70	3.28	4.10	4.78	17
18	0.53	0.90	1.46	2.20	2.77	3.38	4.25	4.96	0.57	0.93	1.45	2.13	2.66	3.22	4.01	4.66	18
19	0.53	0.90	1.46	2.18	2.74	3.33	4.17	4.85	0.57	0.92	1.44	2.11	2.63	3.17	3.94	4.56	19
20	0.53	0.90	1.45	2.16	2.71	3.29	4.10	4.76	0.57	0.92	1.44	2.09	2.60	3.13	3.87	4.47	20
21	0.53	0.90	1.44	2.14	2.68	3.25	4.04	4.68	0.57	0.92	1.43	2.08	2.57	3.09	3.81	4.39	21
22	0.53	0.90	1.44	2.13	2.66	3.22	3.99	4.61	0.57	0.92	1.42	2.06	2.55	3.05	3.76	4.32	22
23	0.53	0.90	1.43	2.11	2.64	3.18	3.94	4.54	0.57	0.92	1.42	2.05	2.53	3.02	3.71	4.26	23
24	0.53	0.90	1.43	2.10	2.62	3.15	3.90	4.49	0.57	0.92	1.41	2.04	2.51	2.99	3.67	4.20	24
25	0.53	0.89	1.42	2.09	2.60	3.13	3.85	4.43	0.57	0.92	1.41	2.02	2.49	2.97	3.63	4.15	25
26	0.53	0.89	1.42	2.08	2.59	3.10	3.82	4.38	0.57	0.91	1.41	2.01	2.47	2.94	3.59	4.10	26
27	0.53	0.89	1.42	2.07	2.57	3.08	3.78	4.34	0.57	0.91	1.40	2.00	2.46	2.92	3.56	4.06	27
28	0.53	0.89	1.41	2.06	2.56	3.06	3.75	4.30	0.57	0.91	1.40	2.00	2.45	2.90	3.53	4.02	28
29	0.53	0.89	1.41	2.06	2.55	3.04	3.73	4.26	0.57	0.91	1.40	1.99	2.43	2.88	3.50	3.98	29
30	0.53	0.89	1.41	2.05	2.53	3.03	3.70	4.23	0.57	0.91	1.39	1.98	2.42	2.87	3.47	3.95	30
40	0.53	0.89	1.39	2.00	2.45	2.90	3.51	3.99	0.57	0.91	1.37	1.93	2.34	2.74	3.29	3.71	40
60	0.53	0.88	1.37	1.95	2.37	2.79	3.34	3.76	0.57	0.90	1.35	1.87	2.25	2.63	3.12	3.49	60
120	0.53	0.88	1.35	1.90	2.29	2.67	3.17	3.55	0.57	0.90	1.33	1.82	2.17	2.52	2.96	3.28	120
∞	0.53	0.87	1.33	1.85	2.21	2.57	3.02	3.35	0.57	0.89	1.31	1.77	2.10	2.41	2.80	3.09	∞
level of significance				10%	5%	2.5%	1%	0.5%				10%	5%	2.5%	1%	0.5%	
				one sided test (upper tail)								one sided test (upper tail)					

TABLE 6.1. (*continued*). THE F DISTRIBUTION: FRACTILES

	$\nu_1 = 7$								$\nu_1 = 8$								
ν_2	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	ν_2
1	0.64	1.98	9.10	58.91	236.8	948.2	5928	23715	0.65	2.00	9.19	59.44	238.9	956.7	5982	23925	1
2	0.59	1.30	3.34	9.35	19.35	39.36	99.36	199.4	0.60	1.32	3.35	9.37	19.37	39.37	99.37	199.4	2
3	0.58	1.15	2.43	5.27	8.89	14.62	27.67	44.43	0.60	1.16	2.44	5.25	8.85	14.54	27.49	44.13	3
4	0.58	1.08	2.08	3.98	6.09	9.07	14.98	21.62	0.60	1.09	2.08	3.95	6.04	8.98	14.80	21.35	4
5	0.58	1.04	1.89	3.37	4.88	6.85	10.46	14.20	0.60	1.05	1.89	3.34	4.82	6.76	10.29	13.96	5
6	0.59	1.02	1.78	3.01	4.21	5.70	8.26	10.79	0.61	1.03	1.78	2.98	4.15	5.60	8.10	10.57	6
7	0.59	1.00	1.70	2.78	3.79	4.99	6.99	8.89	0.61	1.01	1.70	2.75	3.73	4.90	6.84	8.68	7
8	0.59	0.99	1.64	2.62	3.50	4.53	6.18	7.69	0.61	1.00	1.64	2.59	3.44	4.43	6.03	7.50	8
9	0.59	0.98	1.60	2.51	3.29	4.20	5.61	6.88	0.61	0.99	1.60	2.47	3.23	4.10	5.47	6.69	9
10	0.59	0.97	1.57	2.41	3.14	3.95	5.20	6.30	0.61	0.98	1.56	2.38	3.07	3.85	5.06	6.12	10
11	0.59	0.96	1.54	2.34	3.01	3.76	4.89	5.86	0.61	0.98	1.53	2.30	2.95	3.66	4.74	5.68	11
12	0.59	0.96	1.52	2.28	2.91	3.61	4.64	5.52	0.62	0.97	1.51	2.24	2.85	3.51	4.50	5.35	12
13	0.60	0.96	1.50	2.23	2.83	3.48	4.44	5.25	0.62	0.97	1.49	2.20	2.77	3.39	4.30	5.08	13
14	0.60	0.95	1.49	2.19	2.76	3.38	4.28	5.03	0.62	0.96	1.48	2.15	2.70	3.29	4.14	4.86	14
15	0.60	0.95	1.47	2.16	2.71	3.29	4.14	4.85	0.62	0.96	1.46	2.12	2.64	3.20	4.00	4.67	15
16	0.60	0.95	1.46	2.13	2.66	3.22	4.03	4.69	0.62	0.96	1.45	2.09	2.59	3.12	3.89	4.52	16
17	0.60	0.94	1.45	2.10	2.61	3.16	3.93	4.56	0.62	0.96	1.44	2.06	2.55	3.06	3.79	4.39	17
18	0.60	0.94	1.44	2.08	2.58	3.10	3.84	4.44	0.62	0.95	1.43	2.04	2.51	3.01	3.71	4.28	18
19	0.60	0.94	1.43	2.06	2.54	3.05	3.77	4.34	0.62	0.95	1.42	2.02	2.48	2.96	3.63	4.18	19
20	0.60	0.94	1.43	2.04	2.51	3.01	3.70	4.26	0.62	0.95	1.42	2.00	2.45	2.91	3.56	4.09	20
21	0.60	0.94	1.42	2.02	2.49	2.97	3.64	4.18	0.62	0.95	1.41	1.98	2.42	2.87	3.51	4.01	21
22	0.60	0.93	1.41	2.01	2.46	2.93	3.59	4.11	0.62	0.95	1.40	1.97	2.40	2.84	3.45	3.94	22
23	0.60	0.93	1.41	1.99	2.44	2.90	3.54	4.05	0.62	0.95	1.40	1.95	2.37	2.81	3.41	3.88	23
24	0.60	0.93	1.40	1.98	2.42	2.87	3.50	3.99	0.63	0.94	1.39	1.94	2.36	2.78	3.36	3.83	24
25	0.60	0.93	1.40	1.97	2.40	2.85	3.46	3.94	0.63	0.94	1.39	1.93	2.34	2.75	3.32	3.78	25
26	0.60	0.93	1.39	1.96	2.39	2.82	3.42	3.89	0.63	0.94	1.38	1.92	2.32	2.73	3.29	3.73	26
27	0.60	0.93	1.39	1.95	2.37	2.80	3.39	3.85	0.63	0.94	1.38	1.91	2.31	2.71	3.26	3.69	27
28	0.60	0.93	1.39	1.94	2.36	2.78	3.36	3.81	0.63	0.94	1.38	1.90	2.29	2.69	3.23	3.65	28
29	0.60	0.93	1.38	1.93	2.35	2.76	3.33	3.77	0.63	0.94	1.37	1.89	2.28	2.67	3.20	3.61	29
30	0.60	0.93	1.38	1.93	2.33	2.75	3.30	3.74	0.63	0.94	1.37	1.88	2.27	2.65	3.17	3.58	30
40	0.60	0.92	1.36	1.87	2.25	2.62	3.12	3.51	0.63	0.93	1.35	1.83	2.18	2.53	2.99	3.35	40
60	0.60	0.92	1.33	1.82	2.17	2.51	2.95	3.29	0.63	0.93	1.32	1.77	2.10	2.41	2.82	3.13	60
120	0.61	0.91	1.31	1.77	2.09	2.39	2.79	3.09	0.63	0.92	1.30	1.72	2.02	2.30	2.66	2.93	120
∞	0.61	0.91	1.29	1.72	2.01	2.29	2.64	2.90	0.63	0.92	1.28	1.67	1.94	2.19	2.51	2.74	∞
level of significance				10%	5%	2.5%	1%	0.5%				10%	5%	2.5%	1%	0.5%	
				one sided test (upper tail)								one sided test (upper tail)					

TABLE 6.1. *(continued)*. THE F DISTRIBUTION : FRACTILES

	$\nu_1 = 9$								$\nu_1 = 12$								
ν_2	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	ν_2
1	0.66	2.03	9.26	59.86	240.5	963.3	6022	24091	0.68	2.07	9.41	60.71	243.9	976.7	6106	24426	1
2	0.62	1.33	3.37	9.38	19.38	39.39	99.39	199.4	0.64	1.36	3.39	9.41	19.41	39.41	99.42	199.4	2
3	0.61	1.17	2.44	5.24	8.81	14.47	27.35	43.88	0.64	1.20	2.45	5.22	8.74	14.34	27.05	43.39	3
4	0.62	1.10	2.08	3.94	6.00	8.90	14.66	21.14	0.65	1.13	2.08	3.90	5.91	8.75	14.37	20.70	4
5	0.62	1.06	1.89	3.32	4.77	6.68	10.16	13.77	0.65	1.09	1.89	3.27	4.68	6.52	9.89	13.38	5
6	0.62	1.04	1.77	2.96	4.10	5.52	7.98	10.39	0.65	1.06	1.77	2.90	4.00	5.37	7.72	10.03	6
7	0.62	1.02	1.69	2.72	3.68	4.82	6.72	8.51	0.66	1.04	1.68	2.67	3.57	4.67	6.47	8.18	7
8	0.63	1.01	1.63	2.56	3.39	4.36	5.91	7.34	0.66	1.03	1.62	2.50	3.28	4.20	5.67	7.01	8
9	0.63	1.00	1.59	2.44	3.18	4.03	5.35	6.54	0.66	1.02	1.58	2.38	3.07	3.87	5.11	6.23	9
10	0.63	0.99	1.56	2.35	3.02	3.78	4.94	5.97	0.67	1.01	1.54	2.28	2.91	3.62	4.71	5.66	10
11	0.63	0.99	1.53	2.27	2.90	3.59	4.63	5.54	0.67	1.01	1.51	2.21	2.79	3.43	4.40	5.24	11
12	0.63	0.98	1.51	2.21	2.80	3.44	4.39	5.20	0.67	1.00	1.49	2.15	2.69	3.28	4.16	4.91	12
13	0.63	0.98	1.49	2.16	2.71	3.31	4.19	4.94	0.67	1.00	1.47	2.10	2.60	3.15	3.96	4.64	13
14	0.64	0.97	1.47	2.12	2.65	3.21	4.03	4.72	0.67	0.99	1.45	2.05	2.53	3.05	3.80	4.43	14
15	0.64	0.97	1.46	2.09	2.59	3.12	3.89	4.54	0.68	0.99	1.44	2.02	2.48	2.96	3.67	4.25	15
16	0.64	0.97	1.44	2.06	2.54	3.05	3.78	4.38	0.68	0.99	1.43	1.99	2.42	2.89	3.55	4.10	16
17	0.64	0.96	1.43	2.03	2.49	2.98	3.68	4.25	0.68	0.98	1.41	1.96	2.38	2.82	3.46	3.97	17
18	0.64	0.96	1.42	2.01	2.46	2.93	3.60	4.14	0.68	0.98	1.40	1.93	2.34	2.77	3.37	3.86	18
19	0.64	0.96	1.41	1.98	2.42	2.88	3.52	4.04	0.68	0.98	1.40	1.91	2.31	2.72	3.30	3.76	19
20	0.64	0.96	1.41	1.96	2.39	2.84	3.46	3.96	0.68	0.98	1.39	1.89	2.28	2.68	3.23	3.68	20
21	0.64	0.96	1.40	1.95	2.37	2.80	3.40	3.88	0.68	0.98	1.38	1.87	2.25	2.64	3.17	3.60	21
22	0.64	0.96	1.39	1.93	2.34	2.76	3.35	3.81	0.68	0.97	1.37	1.86	2.23	2.60	3.12	3.54	22
23	0.64	0.95	1.39	1.92	2.32	2.73	3.30	3.75	0.68	0.97	1.37	1.84	2.20	2.57	3.07	3.47	23
24	0.64	0.95	1.38	1.91	2.30	2.70	3.26	3.69	0.68	0.97	1.36	1.83	2.18	2.54	3.03	3.42	24
25	0.64	0.95	1.38	1.89	2.28	2.68	3.22	3.64	0.68	0.97	1.36	1.82	2.16	2.51	2.99	3.37	25
26	0.64	0.95	1.37	1.88	2.27	2.65	3.18	3.60	0.69	0.97	1.35	1.81	2.15	2.49	2.96	3.33	26
27	0.64	0.95	1.37	1.87	2.25	2.63	3.15	3.56	0.69	0.97	1.35	1.80	2.13	2.47	2.93	3.28	27
28	0.64	0.95	1.37	1.87	2.24	2.61	3.12	3.52	0.69	0.97	1.34	1.79	2.12	2.45	2.90	3.25	28
29	0.64	0.95	1.36	1.86	2.22	2.59	3.09	3.48	0.69	0.97	1.34	1.78	2.10	2.43	2.87	3.21	29
30	0.64	0.95	1.36	1.85	2.21	2.57	3.07	3.45	0.69	0.97	1.34	1.77	2.09	2.41	2.84	3.18	30
40	0.65	0.94	1.34	1.79	2.12	2.45	2.89	3.22	0.69	0.96	1.31	1.71	2.00	2.29	2.66	2.95	40
60	0.65	0.94	1.31	1.74	2.04	2.33	2.72	3.01	0.69	0.96	1.29	1.66	1.92	2.17	2.50	2.74	60
120	0.65	0.93	1.29	1.68	1.96	2.22	2.56	2.81	0.70	0.95	1.26	1.60	1.83	2.05	2.34	2.54	120
∞	0.65	0.93	1.27	1.63	1.88	2.11	2.41	2.62	0.70	0.95	1.24	1.55	1.75	1.94	2.18	2.36	∞
level of significance				10%	5%	2.5%	1%	0.5%				10%	5%	2.5%	1%	0.5%	
	one sided test (upper tail)								one sided test (upper tail)								

TABLE 6.1. *(continued)*. THE F DISTRIBUTION: FRACTILES

ν_2	$\nu_1 = 24$								$\nu_1 = \infty$								ν_2
	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	p: 0.25	0.50	0.75	0.90	0.95	0.975	0.99	0.995	
1	0.72	2.13	9.63	62.00	249.1	997.2	6235	24940	0.76	2.20	9.85	63.33	254.3	1018	6366	25465	1
2	0.68	1.40	3.43	9.45	19.45	39.46	99.46	199.5	0.72	1.44	3.48	9.49	19.50	39.50	99.50	199.5	2
3	0.68	1.23	2.46	5.18	8.64	14.12	26.60	42.62	0.73	1.27	2.47	5.13	8.53	13.90	26.13	41.83	3
4	0.69	1.16	2.08	3.83	5.77	8.51	13.93	20.03	0.74	1.19	2.08	3.76	5.63	8.26	13.46	19.32	4
5	0.70	1.12	1.88	3.19	4.53	6.28	9.47	12.78	0.75	1.15	1.87	3.10	4.36	6.02	9.02	12.14	5
6	0.71	1.09	1.75	2.82	3.84	5.12	7.31	9.47	0.77	1.12	1.74	2.72	3.67	4.85	6.88	8.88	6
7	0.71	1.07	1.67	2.58	3.41	4.42	6.07	7.65	0.77	1.10	1.65	2.47	3.23	4.14	5.65	7.08	7
8	0.72	1.06	1.60	2.40	3.12	3.95	5.28	6.50	0.78	1.09	1.58	2.29	2.93	3.67	4.86	5.95	8
9	0.72	1.05	1.56	2.28	2.90	3.61	4.73	5.73	0.79	1.08	1.53	2.16	2.71	3.33	4.31	5.19	9
10	0.73	1.04	1.52	2.18	2.74	3.37	4.33	5.17	0.80	1.07	1.48	2.06	2.54	3.08	3.91	4.64	10
11	0.73	1.03	1.49	2.10	2.61	3.17	4.02	4.76	0.80	1.06	1.45	1.97	2.40	2.88	3.60	4.23	11
12	0.73	1.03	1.46	2.04	2.51	3.02	3.78	4.43	0.81	1.06	1.42	1.90	2.30	2.72	3.36	3.90	12
13	0.74	1.02	1.44	1.98	2.42	2.89	3.59	4.17	0.81	1.05	1.40	1.85	2.21	2.60	3.17	3.65	13
14	0.74	1.02	1.42	1.94	2.35	2.79	3.43	3.96	0.82	1.05	1.38	1.80	2.13	2.49	3.00	3.44	14
15	0.74	1.02	1.41	1.90	2.29	2.70	3.29	3.79	0.82	1.05	1.36	1.76	2.07	2.40	2.87	3.26	15
16	0.75	1.01	1.39	1.87	2.24	2.63	3.18	3.64	0.82	1.04	1.34	1.72	2.01	2.32	2.75	3.11	16
17	0.75	1.01	1.38	1.84	2.19	2.56	3.08	3.51	0.82	1.04	1.33	1.69	1.96	2.25	2.65	2.98	17
18	0.75	1.01	1.37	1.81	2.15	2.50	3.00	3.40	0.83	1.04	1.32	1.66	1.92	2.19	2.57	2.87	18
19	0.75	1.01	1.36	1.79	2.11	2.45	2.92	3.31	0.83	1.04	1.30	1.63	1.88	2.13	2.49	2.78	19
20	0.75	1.01	1.35	1.77	2.08	2.41	2.86	3.22	0.84	1.03	1.29	1.61	1.84	2.09	2.42	2.69	20
21	0.75	1.00	1.34	1.75	2.05	2.37	2.80	3.15	0.84	1.03	1.28	1.59	1.81	2.04	2.36	2.61	21
22	0.75	1.00	1.33	1.73	2.03	2.33	2.75	3.08	0.84	1.03	1.28	1.57	1.78	2.00	2.31	2.55	22
23	0.76	1.00	1.33	1.72	2.01	2.30	2.70	3.02	0.85	1.03	1.27	1.55	1.76	1.97	2.26	2.48	23
24	0.76	1.00	1.32	1.70	1.98	2.27	2.66	2.97	0.85	1.03	1.26	1.53	1.73	1.94	2.21	2.43	24
25	0.76	1.00	1.32	1.69	1.96	2.24	2.62	2.92	0.85	1.03	1.25	1.52	1.71	1.91	2.17	2.38	25
26	0.76	1.00	1.31	1.68	1.95	2.22	2.58	2.87	0.86	1.03	1.25	1.50	1.69	1.88	2.13	2.33	26
27	0.76	1.00	1.31	1.67	1.93	2.19	2.55	2.83	0.86	1.03	1.24	1.49	1.67	1.85	2.10	2.29	27
28	0.76	1.00	1.30	1.66	1.91	2.17	2.52	2.79	0.86	1.02	1.24	1.48	1.65	1.83	2.06	2.25	28
29	0.76	1.00	1.30	1.65	1.90	2.15	2.49	2.76	0.86	1.02	1.23	1.47	1.64	1.81	2.03	2.21	29
30	0.76	0.99	1.29	1.64	1.89	2.14	2.47	2.73	0.86	1.02	1.23	1.46	1.62	1.79	2.01	2.18	30
40	0.77	0.99	1.26	1.57	1.79	2.01	2.29	2.50	0.88	1.02	1.19	1.38	1.51	1.64	1.80	1.93	40
60	0.78	0.98	1.24	1.51	1.70	1.88	2.12	2.29	0.90	1.01	1.15	1.29	1.39	1.48	1.60	1.69	60
120	0.78	0.98	1.21	1.45	1.61	1.76	1.95	2.09	0.92	1.01	1.10	1.19	1.25	1.31	1.38	1.43	120
∞	0.79	0.97	1.18	1.38	1.52	1.64	1.79	1.90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	∞
level of significance				10%	5%	2.5%	1%	.5%				10%	5%	2.5%	1%	.5%	
				one sided test (upper tail)								one sided test (upper tail)					

6.2. Beta Function Representation

a. Introduction

Table 6.2 gives upper 1% and 5% values of the beta distribuion with the density

$$\frac{1}{B(a, b)}\ u^{a-1}(1-u)^{b-1}, 0 \leqslant u \leqslant 1,$$

for the following values of the parameters a and b :

$$2a = 1(1)\ 9,\ 12,\ 24,\ \infty$$

$$2b = 1(1)30,\ 40,\ 60,\ 120,\ \infty.$$

Fractiles corresponding to $p = 0.01$, 0.05 (the lower 1% and 5% points) can be read from Table 6.2 by interchanging a and b and taking the difference from unity of the table entry.

Example. To find the fractile for $2a = 5$, $2b = 7$, and $p = 0.05$. The required fractile is $1-0.87222 = 0.12778$, 0.87222 being the upper 5% point (0.95th fractile) of beta with $2a = 7$, $2b = 5$.

b. Beta distribution—its relation to the distribution of the variance ratio (F) and the null-distribution of the multiple correlation coefficient

Consider the two transformations

$$(1)\quad u = \frac{\nu_1 F}{\nu_2+\nu_1 F}$$

$$(2)\quad u = \frac{\nu_2}{\nu_2+\nu_1 F}\ .$$

The first equation transforms the variance ratio (F) having parameters ν_1 and ν_2 (d.f. of numerator and denominator) to a beta variable having parameters $a = \frac{\nu_1}{2}$, $b = \frac{\nu_2}{2}$ while the second transforms the same variance ratio to a beta variable with parameters a and b interchanged i.e. : $a = \frac{\nu_2}{2}$, $b = \frac{\nu_1}{2}$. Table 6.2 directly gives the significant values of R^2 the square of the multiple correlation coefficient with $2a = k$, the number of independent variables and $2b = n-k-1$, where n is the sample size. In **6.1c** the significance of R^2 was judged by first computing a function of R^2 which is distributed as F and referring to the F table.

c. The incomplete beta function—its relation to cumulated binomial probabilities

An equation connecting the incomplete beta integral with the cumulative sum of binomial probabilities is given in **1.3b**. The use of Table 6.2 in determining one-sided confidence limits to the parameter π of the binomial distribution and in providing one sided tests of hypothesis concerning π is already demonstrated in **1.3b and c.**

TABLE 6.2. THE BETA DISTRIBUTION
(Upper 1% values)

2b \ 2a	1	2	3	4	5	6	7	8	9	12	24	∞
1	.99975	.99990	.99994	.99996	.99997	.99997	.99998	.99998	.99998	.99999	.99999	1
2	.98010	.99000	.99332	.99499	.99599	.99666	.99713	.99749	.99777	.99833	.99916	1
3	.91917	.95358	.96717	.97454	.97919	.98240	.98475	.98654	.98796	.99084	.99532	1
4	.84125	.90000	.92604	.94110	.95099	.95800	.96325	.96732	.97057	.97734	.98818	1
5	.76480	.84151	.87858	.90112	.91644	.92757	.93605	.94274	.94814	.95957	.97847	1
6	.69013	.78456	.83021	.85913	.87935	.89436	.90599	.91527	.92286	.93916	.96694	1
7	.63630	.73173	.78364	.81764	.84199	.86041	.87489	.88659	.89625	.91729	.95418	1
8	.58460	.68377	.74003	.77793	.80563	.82693	.84388	.85773	.86927	.89474	.94061	1
9	.53991	.64062	.69976	.74055	.77090	.79457	.81363	.82934	.84255	.87204	.92653	1
10	.50111	.60189	.66281	.70569	.73809	.76368	.78449	.80180	.81645	.84956	.91216	1
11	.46721	.56712	.62901	.67333	.70729	.73440	.75665	.77531	.79121	.82750	.89768	1
12	.43742	.53584	.59809	.64336	.67847	.70677	.73019	.74997	.76693	.80602	.88319	1
13	.41107	.50761	.56980	.61563	.65155	.68076	.70513	.72583	.74369	.78521	.86880	1
14	.38762	.48205	.54385	.58994	.62642	.65631	.68142	.70288	.72149	.76511	.85456	1
15	.36664	.45883	.52001	.56613	.60294	.63334	.65902	.68109	.70032	.74574	.84052	1
16	.34776	.43766	.49806	.54403	.58101	.61174	.63786	.66042	.68015	.72711	.82673	1
17	.33070	.41829	.47781	.52349	.56049	.59143	.61787	.64080	.66095	.70921	.81319	1
18	.31521	.40052	.45906	.50435	.54128	.57232	.59897	.62219	.64267	.69203	.79995	1
19	.30108	.38415	.44168	.48650	.52326	.55432	.58110	.60453	.62526	.67554	.78699	1
20	.28815	.36904	.42553	.46982	.50634	.53734	.56419	.58776	.60869	.65971	.77433	1
21	.27628	.35505	.41048	.45419	.49042	.52132	.54816	.57182	.59289	.64452	.76197	1
22	.26533	.34207	.39643	.43954	.47644	.50617	.53297	.55667	.57783	.62995	.74992	1
23	.25521	.32998	.38329	.42578	.46131	.49184	.51856	.54225	.56347	.61595	.73816	1
24	.24583	.31871	.37097	.41283	.44796	.47826	.50486	.52851	.54975	.60251	.72671	1
25	.23710	.30817	.35941	.40062	.43534	.46539	.49184	.51542	.53665	.58960	.71554	1
26	.22897	.29830	.34853	.38910	.42340	.45317	.47945	.50294	.52413	.57720	.70466	1
27	.22138	.28903	.33828	.37820	.41207	.44155	.46764	.49101	.51215	.56527	.69406	1
28	.21427	.28031	.32861	.36789	.40132	.43049	.45638	.47962	.50068	.55379	.68374	1
29	.20760	.27210	.31946	.35812	.39110	.41996	.44563	.46873	.48969	.54274	.67368	1
30	.20133	.26436	.31081	.34884	.38138	.40992	.43536	.45830	.47915	.53211	.66388	1
40	.15459	.20567	.24439	.27684	.30518	.33050	.35344	.37445	.39383	.44427	.57856	1
60	.10551	.14230	.17102	.19567	.21767	.23773	.25624	.27349	.28966	.33299	.45833	1
120	.05401	.07388	.08986	.10393	.11679	.12876	.14005	.15076	.16100	.18938	.28058	1
∞	0	0	0	0	0	0	0	0	0	0	0	

The table gives the values of x for which $\int_x^1 u^{a-1}(1-u)^{b-1}du/B(a, b) = 0.01$

TABLE 6.2. *(continued)*. THE BETA DISTRIBUTION
(Upper 5% values)

2b \ 2a	1	2	3	4	5	6	7	8	9	12	24	∞
1	.99384	.99750	.99846	.99889	.99913	.99929	.99940	.99948	.99954	.99966	.99983	1
2	.90250	.95000	.96638	.97468	.97969	.98305	.98545	.98726	.98867	.99149	.99573	1
3	.77148	.86428	.90269	.92399	.93759	.94704	.95399	.95933	.96355	.97221	.98574	1
4	.65837	.77639	.83175	.86465	.88662	.90239	.91427	.92356	.93102	.94663	.97195	1
5	.56926	.69829	.76447	.80597	.83472	.85592	.87222	.88518	.89573	.91821	.95601	1
6	.49947	.63160	.70401	.75140	.78523	.81074	.83073	.84684	.86011	.88889	.93890	1
7	.44407	.57511	.65071	.70189	.73937	.76818	.79110	.80981	.82539	.85971	.92122	1
8	.39929	.52713	.60393	.65741	.69740	.72866	.75387	.77468	.79217	.83125	.90334	1
9	.36249	.48610	.56284	.61755	.65920	.69223	.71918	.74165	.76070	.80382	.88551	1
10	.33176	.45072	.52662	.58180	.62447	.65874	.68699	.71076	.73106	.77756	.86789	1
11	.30575	.41997	.49454	.54967	.59288	.62797	.65717	.68193	.70323	.75254	.85057	1
12	.28346	.39304	.46598	.52070	.56410	.59969	.62956	.65506	.67714	.72875	.83364	1
13	.26417	.36927	.44042	.49449	.53781	.57365	.60396	.63000	.65268	.70617	.81712	1
14	.24732	.34816	.41744	.47068	.51374	.54964	.58020	.60662	.62975	.68476	.80105	1
15	.23246	.32930	.39667	.44898	.49164	.52745	.55813	.58479	.60824	.66446	.78543	1
16	.21928	.31234	.37783	.42914	.47128	.50690	.53758	.56437	.58804	.64520	.77028	1
17	.20751	.29703	.36067	.41093	.45250	.48783	.51842	.54526	.56906	.62693	.75559	1
18	.19693	.28313	.34497	.39416	.43510	.47009	.50051	.52733	.55120	.60959	.74135	1
19	.18737	.27046	.33056	.37869	.41897	.45355	.48376	.51049	.53436	.59311	.72756	1
20	.17869	.25887	.31729	.36436	.40395	.43811	.46806	.49465	.51848	.57744	.71420	1
21	.17077	.24822	.30504	.35106	.38996	.42365	.45331	.47973	.50348	.56254	.70126	1
22	.16353	.23840	.29368	.33868	.37688	.41010	.43944	.46566	.48929	.54835	.68874	1
23	.15687	.22933	.28313	.32713	.36464	.39737	.42637	.45236	.47585	.53482	.67660	1
24	.15073	.22092	.27331	.31634	.35316	.38539	.41404	.43978	.46311	.52192	.66485	1
25	.14506	.21310	.26414	.30623	.34236	.37410	.40239	.42787	.45102	.50960	.65347	1
26	.13979	.20582	.25556	.29673	.33220	.36344	.39136	.41657	.43952	.49783	.64244	1
27	.13480	.19901	.24751	.28781	.32262	.35337	.38091	.40584	.42859	.48657	.63174	1
28	.13033	.19264	.23996	.27940	.31357	.34383	.37100	.39564	.41817	.47580	.62138	1
29	.12606	.18666	.23285	.27146	.30501	.33478	.36158	.38593	.40823	.46548	.61133	1
30	.12206	.18104	.22614	.26396	.29689	.32619	.35262	.37668	.39875	.45558	.60158	1
40	.09266	.13911	.17553	.20673	.23441	.25947	.28242	.30364	.32337	.37540	.51825	1
60	.06252	.09503	.12119	.14409	.16483	.18394	.20176	.21850	.23431	.27718	.40478	1
120	.03163	.04870	.06280	.07542	.08710	.09808	.10852	.11850	.12809	.15496	.24339	1
∞	0	0	0	0	0	0	0	0	0	0	0	

The table gives the values of x for which $\int_x^1 u^{a-1}(1-u)^{b-1}du/B(a,b) = 0.05$

6.3. The Distribution of s^2_{max}/s^2_{min}

a. Introduction

Table 6.3 gives the upper 1% and 5% values of s^2_{max}/s^2_{min} where s^2_{max} and s^2_{min} are respectively the largest and the smallest in a set of k independent mean squares each based on ν d.f.

b. Application

Seven pieces of yarn were sampled from each of 5 spinning frames and tested for tensile strength. The values of s^2 for the 5 frames are 0.0297, 0.0429, 0.0381, 0.1181, 0.0467. To test whether the variability is the same for all frames, compute : $s^2_{max}/s^2_{min} = 0.1181/0.0297 = 3.98$. The 5% value of s^2_{max}/s^2_{min} for $k = 5$ and $\nu = 6$ is 12.1, so that the observed ratio is not significant at the 5% level.

TABLE 6.3. UPPER PERCENTAGE POINTS OF s^2_{max}/s^2_{min}

(Upper 1% points)

ν \ k	2	3	4	5	6	7	8	9	10	11	12
2	199	448	729	1036	1362	1705	2063	2432	2813	3204	3605
3	47.5	85	120	151	184	21(6)	24(9)	28(1)	31(0)	33(7)	36(1)
4	23.2	37	49	59	69	79	89	97	106	113	120
5	14.9	22	28	33	38	42	46	50	54	57	60
6	11.1	15.5	19.1	22	25	27	30	32	34	36	37
7	8.89	12.1	14.5	16.5	18.4	20	22	23	24	26	27
8	7.50	9.9	11.7	13.2	14.5	15.8	16.9	17.9	18.9	19.8	21
9	6.54	8.5	9.9	11.1	12.1	13.1	13.9	14.7	15.3	16.0	16.6
10	5.85	7.4	8.6	9.6	10.4	11.1	11.8	12.4	12.9	13.4	13.9
12	4.91	6.1	6.9	7.6	8.2	8.7	9.1	9.5	9.9	10.2	10.6
15	4.07	4.9	5.5	6.0	6.4	6.7	7.1	7.3	7.5	7.8	8.0
20	3.32	3.8	4.3	4.6	4.9	5.1	5.3	5.5	5.6	5.8	5.9
30	2.63	3.0	3.3	3.4	3.6	3.7	3.8	3.9	4.0	4.1	4.2
60	1.96	2.2	2.3	2.4	2.4	2.5	2.5	2.6	2.6	2.7	2.7
∞	1.00	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

(Upper 5% points)

ν \ k	2	3	4	5	6	7	8	9	10	11	12
2	39.0	87.5	142	202	266	333	403	475	550	626	704
3	15.4	27.8	39.2	50.7	62.0	72.9	83.5	93.9	104	114	124
4	9.60	15.5	20.6	25.2	29.5	33.6	37.5	41.1	44.6	48.0	51.4
5	7.15	10.8	13.7	16.3	18.7	20.8	22.9	24.7	26.5	28.2	29.9
6	5.82	8.38	10.4	12.1	13.7	15.0	16.3	17.5	18.6	19.7	20.7
7	4.99	6.94	8.44	9.70	10.8	11.8	12.7	13.5	14.3	15.1	15.8
8	4.43	6.00	7.18	8.12	9.03	9.78	10.5	11.1	11.7	12.2	12.7
9	4.03	5.34	6.31	7.11	7.80	8.41	8.95	9.45	9.91	10.3	10.7
10	3.72	4.85	5.67	6.34	6.92	7.42	7.87	8.28	8.66	9.01	9.34
12	3.28	4.16	4.79	5.30	5.72	6.09	6.42	6.72	7.00	7.25	7.48
15	2.86	3.54	4.01	4.37	4.68	4.95	5.19	5.40	5.59	5.77	5.93
20	2.46	2.95	3.29	3.54	3.76	3.94	4.10	4.24	4.37	4.49	4.59
30	2.07	2.40	2.61	2.78	2.91	3.02	3.12	3.21	3.29	3.36	3.39
60	1.67	1.85	1.96	2.04	2.11	2.17	2.22	2.26	2.30	2.33	2.36
∞	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Values in the column $k = 2$ and in the rows $\nu = 2$ and ∞ are exact. Elsewhere the third digit may be in error by a few units for the 5% points and several units for the 1% points. The third digit figures in brackets for $\nu = 3$ are the most uncertain.

7. THE CORRELATION COEFFICIENT

In 4c was described a t test for testing the significance of an observed sample correlation coefficient. Table 7.1 gives directly the significant values of r (the sample total or partial correlation coefficient under the assumption of normality) correct to three places of decimal and d.f. = 1(1) 30 (10) 80, 100(50) 300. With this table, the computation of t for testing the significance of r, is unnecessary.

The first three columns give 5%, 1% and 0.1% level values of $|r|$ for two sided tests. The next three columns give upper tail values for one sided tests at 5%, 1% and 0.1% levels of significance. A negative sign prefixed to these upper tail critical values provides the corresponding lower tail values.

Example : The value of the sample correlation coefficient between head length and head breadth computed from measurements on 30 individuals is 0.415. To test the hypothesis that the population correlation coefficient is zero.

Here the d.f. is 30–2 = 28 and the 5% tabulated value for 28 d.f. is 0.361 for a two sided test. The observed value being larger, the result is significant at the 5% level. If it is known apriori that under the alternative hypothesis the population correlation coefficient would be positive, a one sided test is used for judging the significance of the observed correlation coefficient. The 5% tabulated value for one sided test is 0.306, thus establishing significance of the observed correlation coefficient.

TABLE 7.1 THE CRITICAL VALUES OF THE CORRELATION COEFFICIENT (TOTAL OR PARTIAL)

5%, 1% and 0.1% values for one-sided (upper tail) and two-sided tests

	two-sided			one-sided				two-sided			one-sided		
d.f.	5%	1%	0.1%	5%	1%	0.1%	d.f.	5%	1%	0.1%	5%	1%	0.1%
1	$.9^{2}69$	$.9^{3}88$	$.9^{5}88$	.988	$.9^{3}51$	$.9^{5}51$	21	.413	.526	.640	.352	.482	.610
2	.950	$.9^{2}00$	$.9^{3}00$	.900	.980	$.9^{2}80$	22	.404	.515	.629	.344	.472	.599
3	.878	.959	$.9^{2}11$	.805	.934	.986	23	.396	.505	.618	.337	.462	.588
4	.811	.917	.974	.729	.882	.963	24	.388	.496	.607	.330	.453	.578
5	.754	.875	.951	.669	.833	.935	25	.381	.487	.597	.323	.445	.568
6	.707	.834	.925	.621	.789	.905	26	.374	.478	.588	.317	437	.559
7	.666	.798	.898	.582	.750	.875	27	.367	.470	.579	.311	.430	.550
8	.632	.765	.872	.549	.715	.847	28	.361	.463	.570	.306	.423	.541
9	.602	.735	.847	.521	.685	.820	29	.355	.456	.562	.301	.416	.533
10	.576	.708	.823	.497	.658	.795	30	.349	.449	554	.296	.409	.526
11	.553	.684	.801	.476	.634	.772	40	.304	.393	.490	.257	.358	.463
12	.532	.661	.780	.457	.612	.750	50	.273	.354	.443	.231	.322	.419
13	.514	.641	.760	.441	.592	.730	60	.250	.325	.408	.211	.295	.385
14	.497	.623	.742	.426	.574	.711	70	.232	.302	.380	.195	.274	.358
15	.482	.606	.725	.412	.558	.694	80	.217	.283	.357	.183	.257	.336
16	.468	.590	.708	.400	.543	.678	100	.195	.254	.321	.164	.230	.302
17	.456	.575	.693	.389	.529	.662	150	.159	.208	.263	.134	.189	.249
18	.444	.561	.679	.378	.516	.648	200	.138	.181	.230	.116	.164	.216
19	.433	.549	.665	.369	.503	.635	250	.124	.162	.206	.104	.146	.194
20	.423	.537	.652	.360	.492	.622	300	.113	.146	.188	.095	.134	.177

Note that for testing the significance of correlation coefficient (total) computed form n pairs of observations, the appropriate degrees of freedom are n–2. For testing the significance of a partial correlation coefficient between two variables eliminating k independent variables, computed from observations on n individuals (i.e. on n sets of observations) the degrees of freedom are n–2–k. Thus the partial correlation coefficient $r_{12.3456}$ = 0.63 based on 30 observations is significant against the 5% critical value on 24 d.f.

8. TRANSFORMATIONS

8.1. The $\text{Sin}^{-1}\sqrt{p}$ Transformation For The Binomial Proportion

Introduction

Table 8.1 gives the values of $\sin^{-1}\sqrt{p}$ (in degrees, correct to 3 places of decimal) for $p = 0.000(0.001)0.200\ (0.005)0.500$. For $0.500 < p < 1$, use the formula $\sin^{-1}\sqrt{p} = 90 - \sin^{-1}\sqrt{1-p}$. Thus

$$\sin^{-1}\sqrt{0.785} = 90 - \sin^{-1}\sqrt{0.215} = 90 - 27.625 = 62.375.$$

b. Interpolation in Table 8.1.

For interpolation within the interval 0.000 to 0.030 use the formula $\sin^{-1}\sqrt{p} = 57.29578\sqrt{p}(1+p/6)$ degrees. Linear interpolation should suffice in the interval $(0.030 - 0.500)$. To facilitate linear interpolation within the interval 0.200 to 0.500, values of $\Delta'(= 200\Delta)$ have also been provided in an adjacent column, the formula applicable being

$$\sin^{-1}\sqrt{p} = \sin^{-1}\sqrt{p_0} + \Delta'(p - p_0)$$

where p_0 is the nearest tabular argument below p. Thus

$$\sin^{-1}\sqrt{0.3035} = \sin^{-1}\sqrt{0.300} + \Delta'(0.0035) = 33.211 + 62.4(0.0035) = 33.429$$

observing that the tabulated value of Δ' for $p = 0.300$ is 62.4

c. Application

The binomial proportion x/n has mean π and standard deviation $[\pi(1-\pi)/n]^{\frac{1}{2}}$, but the standard error of $\sin^{-1}\sqrt{p}$ (expressed in degrees as in Table 8.1) is independent of π and is equal to $28.64789/\sqrt{n}$ degrees. Because of this there is some theoretical advantage in transforming an observed proportion p to $\sin^{-1}\sqrt{p}$ in the comparison of proportions in one or multiple way classification by analysis of variance.

The table is also useful in evaluating the inverse of other trigonometric functions.

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}, \qquad \operatorname{cosec}^{-1}x = \sin^{-1}(1/x),$$
$$\tan^{-1}x = \sin^{-1}\sqrt{x^2/(1+x^2)}, \quad \sec^{-1}x = \sin^{-1}\sqrt{(x^2-1)/x^2},$$
$$\cot^{-1}x = \sin^{-1}\sqrt{1/(1+x^2)}.$$

Thus $\tan^{-1}1.24 = \sin^{-1}\sqrt{0.6059} = 90 - \sin^{-1}\sqrt{1-0.6059} = 90 - \sin^{-1}\sqrt{0.3941}$.

$$= 90 - 38.886 = 51.114$$

using the table to find $\sin^{-1}\sqrt{0.3941}$.

d. Some other tables

1. Snedecor, G. W. (1946) : *Statistical Methods*, 4th Ed., Iowa State Univ. Press, Ames, Iowa,
   gives $\text{Sin}^{-1}\sqrt{p}$ correct to two places of decimal for $p = 0(0.0001)\ .01(.001)\ .99(.0001)1$.

2. Fisher, R. A. and Yates, F. (1957) : *Statistical Tables for Biological, Agricultural and Medical Research*, 5th edition, Oliver and Boyd, London,
   gives $\text{Sin}^{-1}\sqrt{p}$ correct to one place of decimal for $p = 0(0.01)\ 0.99$ (Table X) and also for $p = x/n$; $x = 1(1)\ [\frac{1}{2}n]$, $n = 2(1)\ 30$ (Table XI).

3. Hald, A. (1952) : *Statistical Tables and Formulas*, John Wiley and Sons, New York.
   Table 12 gives $2\ \text{Sin}^{-1}\sqrt{p}$ in radians, correct to four places of decimal for $p = 0(0.001)\ 1.000$.

TABLE 8.1. THE $\mathrm{SIN}^{-1}\sqrt{p}$ TRANSFORMATION FOR THE BINOMIAL PROPORTION

Transformation from proportions to degrees

$p = 0.000(0.001)0.199$

p	0	1	2	3	4	5	6	7	8	9
.00	.000	1.812	2.563	3.140	3.626	4.055	4.442	4.799	5.132	5.444
.01	5.739	6.020	6.289	6.547	6.795	7.035	7.267	7.492	7.710	7.923
.02	8.130	8.329	8.530	8.723	8.912	9.098	9.279	9.457	9.632	9.805
.03	9.974	10.141	10.305	10.466	10.626	10.783	10.937	11.090	11.241	11.390
.04	11.537	11.682	11.826	11.968	12.108	12.247	12.385	12.521	12.656	12.789
.05	12.921	13.052	13.181	13.310	13.437	13.563	13.689	13.813	13.936	14.058
.06	14.179	14.299	14.418	14.537	14.654	14.771	14.886	15.001	15.116	15.229
.07	15.342	15.454	15.565	15.675	15.785	15.894	16.003	16.110	16.217	16.324
.08	16.430	16.535	16.640	16.744	16.847	16.951	17.053	17.155	17.256	17.357
.09	17.457	17.557	17.657	17.756	17.854	17.952	18.049	18.147	18.243	18.339
.10	18.435	18.530	18.625	18.719	18.814	18.907	19.001	19.093	19.186	19.278
.11	19.370	19.461	19.552	19.643	19.733	19.823	19.913	20.002	20.091	20.180
.12	20.268	20.356	20.444	20.531	20.618	20.705	20.791	20.877	20.963	21.049
.13	21.134	21.219	21.304	21.389	21.473	21.557	21.641	21.724	21.807	21.890
.14	21.973	22.055	22.137	22.219	22.301	22.383	22.464	22.545	22.626	22.706
.15	22.786	22.867	22.946	23.026	23.106	23.185	23.264	23.343	23.421	23.500
.16	23.578	23.656	23.734	23.812	23.889	23.966	24.044	24.121	24.197	24.274
.17	24.350	24.426	24.502	24.578	24.654	24.729	24.804	24.880	24.955	25.029
.18	25.104	25.179	25.253	25.327	25.401	25.475	25.549	25.622	25.696	25.769
.19	25.842	25.915	25.988	26.060	26.133	26.205	26.277	26.349	26.421	26.493

$p = 0.200(0.005)0.500$

p	$\mathrm{Sin}^{-1}\sqrt{p}$	Δ'	p	$\mathrm{Sin}^{-1}\sqrt{p}$	Δ'	p	$\mathrm{Sin}^{-1}\sqrt{p}$	Δ'
.200	26.565	71.4	.300	33.211	62.4	.400	39.231	58.6
.205	26.922	70.6	.305	33.523	62.0	.405	39.524	58.2
.210	27.275	70.0	.310	33.833	61.8	.410	39.815	58.2
.215	27.625	69.4	.315	34.142	61.6	.415	40.106	58.2
.220	27.972	68.8	.320	34.450	61.2	.420	40.397	58.0
.225	28.316	68.4	.325	34.756	61.2	.425	40.687	57.8
.230	28.658	67.8	.330	35.062	60.8	.430	40.976	57.8
.235	28.997	67.4	.335	35.366	60.6	.435	41.265	57.8
.240	29.334	66.8	.340	35.669	60.2	.440	41.554	57.6
.245	29.668	66.4	.345	35.970	60.2	.445	41.842	57.6
.250	30.000	66.0	.350	36.271	60.0	.450	42.130	57.6
.255	30.330	65.4	.355	36.571	59.8	.455	42.418	57.6
.260	30.657	65.2	.360	36.870	59.6	.460	42.706	57.4
.265	30.983	64.6	.365	37.168	59.4	.465	42.993	57.4
.270	31.306	64.4	.370	37.465	59.2	.470	43.280	57.4
.275	31.628	64.0	.375	37.761	59.2	.475	43.567	57.4
.280	31.948	63.6	.380	38.057	58.8	.480	43.854	57.4
.285	32.266	63.4	.385	38.351	59.0	.485	44.141	57.2
.290	32.583	62.8	.390	38.646	58.6	.490	44.427	57.4
.295	32.897	62.8	.395	38.939	58.4	.495	44.714	57.2
						.500	45.000	

Interpolation in Table 8.1

For $p < 0.03$, use the formula $\sin^{-1}\sqrt{p} = 57.29578(1+p/6)\sqrt{p}$. Linear interpolation would suffice elsewhere. For $0.03 < p < 0.20$ if p_0 and p_1 be two consecutive arguments in the first table such that $p_0 < p < p_1$, use the formula $\sin^{-1}\sqrt{p} = 10^3[(p_1-p)\sin^{-1}\sqrt{p_1}+(p-p_0)\sin^{-1}\sqrt{p_0}]$. For $p > 0.20$ the values of Δ' given in Table 8.1 could be used in the following formula for linear interpolation

$$\sin^{-1}\sqrt{p} = \sin^{-1}\sqrt{p_0} + \Delta'(p-p_0)$$

where p_0 is the nearest tabular argument below p. For $p > .500$, use the formula $\sin^{-1}\sqrt{p} = 90-\sin^{-1}\sqrt{1-p}$.

8.2. The $\tanh^{-1}$ Transformation For Correlation Coefficient

a. Introduction

Table 8.2 gives the values of $z = \tanh^{-1} r = \frac{1}{2} \log_e \frac{1+r}{1-r}$ correct to five places of decimal for $r = 0.00(0.02)\,0.20(0.002)\,0.860(0.001)\,0.999$.

b. Interpolation in Table 8.2

Within the interval $0.20 < r < 0.95$, linear interpolation gives accuracy to four places of decimal For $0 < r < 0.20$ the formula

$$\tanh^{-1} r = r + \frac{r^3}{3}$$

could be used. For $0.95 < r \leqslant 0.99$ quadratic interpolation is necessary to achieve the same degree of accuracy. Interpolation in the table is not advisable for values of $r > 0.99$. In such a case one should compute $\tanh^{-1} r$ directly using the formula

$$z = \tanh^{-1} r = \frac{1}{2} \log_e \frac{1+r}{1-r}.$$

c. Application

(i) *The product moment correlation coefficient* (interclass correlation)

For the sample correlation coefficient r in a sample of size n from the bivariate normal population,

$$E(r) = \rho \left[1 - \frac{1}{2n} - \frac{3}{8n^2} + \rho^2 \left(\frac{1}{2n} - \frac{3}{4n^2}\right) + \rho^4 \frac{9}{8n^2}\right] + \ldots$$

$$\sim \rho\left[1 - \frac{1-\rho^2}{2(n-1)} \left\{1 - \frac{1}{4(n-1)} (1 - 9\rho^2)\right\}\right]$$

and variance

$$V(r) = \frac{1}{n}(1-\rho^2)^2 \left(1 + \frac{1}{n} + \frac{11\rho^2}{2n}\right) + \ldots$$

$$\sim \left[\frac{1-\rho^2}{\sqrt{n-1}} \left\{1 + \frac{11\rho^2}{4(n-1)}\right\}\right]^2$$

where ρ is the population correlation coefficient. For large n,

$$\zeta = E(z) = \tanh^{-1} \rho + \frac{\rho}{2(n-1)} + \ldots \sim \tanh^{-1} \rho$$

and

$$V(z) \sim \frac{1}{n-3}.$$

The same formulae for expectation and variance hold good for a partial correlation coefficient with n changed to $n-p$ where p is the number of variables eliminated.

d. The intraclass correlation coefficient

For the intraclass correlation coefficient r, based on k variates within a class, Fisher proposed the transformation $z = \frac{1}{2}\log_e \frac{1+(k-1)r}{1-r}$ The transformed value in this case may be obtained by first computing $r' = \frac{kr}{2+(k-2)r}$ and reading the value of $\tanh^{-1} r'$ from Table 8.2.

For a given value of $\frac{1}{2}\log_e \frac{1+(k-1)r}{1-r} = c$ the corresponding value of r may be obtained in a similar manner by first obtaining the value of $r' = \tanh c$ by inverse interpolation in Table 8.2 and computing

$$r = \frac{2r'}{2r'+k(1-r')}$$

The expected value and variance of z, in sampling from a normal population, are given by

$$E(z) \sim \frac{1}{2}\log_e \frac{1+(k-1)\rho}{1-\rho}$$

$$V(z) \sim k/2(k-1)(n-2)$$

The transformation to z would be useful in testing for an assigned value of the correlation coefficient (total, partial or intra-class) or in testing the equality of k correlation coefficients on the basis of estimates.

e. Another table

HARVARD UNIVERSITY COMPUTATION LABORATORY (1949): *Tables of Inverse Hyperbolic Functions*, The Annals of the Computation Laboratory of Harvard University, 20, Harvard Univ. Press, Cambridge (Massachusetts)

gives $\tanh^{-1} x$ to 9 places of decimal for $x = 0(0.001)\ 0.5\ (0.0005)\ 0.75\ (0.0002)\ 0.9\ (0.0001)\ 0.95\ (0.00005)\ 0.975\ (0.00002)\ 0.99\ (0.00001)\ 0.99999$.

TABLE 8.2. THE TANH⁻¹ TRANSFORMATION FOR CORRELATION COEFFICIENT

$r = 0.00(0.02)0.18$

r	0	2	4	6	8
.0	.00000	.02000	.04002	.06007	.08017
.1	.10034	.12058	.14093	.16139	.18198

$r = 0.200(0.002)0.858$

r	0	2	4	6	8
.20	.20273	.20482	.20690	.20899	.21108
.21	.21317	.21526	.21736	.21946	.22156
.22	.22366	.22576	.22786	.22997	.23208
.23	.23419	.23630	.23842	.24053	.24265
.24	.24477	.24690	.24902	.25115	.25328
.25	.25541	.25755	.25968	.26182	.26396
.26	.26611	.26825	.27040	.27255	.27471
.27	.27686	.27902	.28118	.28335	.28551
.28	.28768	.28985	.29203	.29420	.29638
.29	.29857	.30076	.30294	.30513	.30732
.30	.30952	.31172	.31392	.31613	.31833
.31	.32055	.32276	.32498	.32720	.32942
.32	.33165	.33388	.33611	.33835	.34059
.33	.34283	.34507	.34732	.34958	.35183
.34	.35409	.35636	.35862	.36089	.36317
.35	.36544	.36772	.37001	.37230	.37459
.36	.37689	.37919	.38149	.38380	.38611
.37	.38842	.39074	.39307	.39539	.39772
.38	.40006	.40240	.40474	.40709	.40944
.39	.41180	.41416	.41653	.41890	.42127
.40	.42365	.42603	.42842	.43081	.43321
.41	.43561	.43802	.44043	.44285	.44527
.42	.44769	.45012	.45256	.45500	.45745
.43	.45990	.46235	.46481	.46728	.46975
.44	.47223	.47471	.47720	.47970	.48220
.45	.48470	.48721	.48973	.49225	.49478
.46	.49731	.49985	.50240	.50495	.50751
.47	.51007	.51264	.51522	.51780	.52039
.48	.52298	.52559	.52819	.53081	.53343
.49	.53606	.53870	.54134	.54399	.54664
.50	.54931	.55198	.55465	.55734	.56003
.51	.56273	.56544	.56815	.57087	.57360
.52	.57634	.57908	.58184	.58460	.58737
.53	.59015	.59293	.59572	.59853	.60134
.54	.60416	.60698	.60982	.61266	.61552

r	0	2	4	6	8
.55	.61838	.62125	.62413	.62702	.62992
.56	.63283	.63575	.63868	.64162	.64457
.57	.64752	.65049	.65347	.65646	.65945
.58	.66246	.66548	.66851	.67155	.67460
.59	.67767	.68074	.68382	.68692	.69003
.60	.69315	.69628	.69942	.70258	.70574
.61	.70892	.71211	.71532	.71853	.72176
.62	.72501	.72826	.73153	.73481	.73811
.63	.74142	.74474	.74808	.75143	.75479
.64	.75817	.76157	.76498	.76840	.77184
.65	.77530	.77877	.78226	.78576	.78928
.66	.79281	.79637	.79993	.80352	.80712
.67	.81074	.81438	.81804	.82171	.82540
.68	.82911	.83284	.83659	.84036	.84415
.69	.84796	.85178	.85563	.85950	.86339
.70	.86730	.87123	.87519	.87916	.88316
.71	.88718	.89123	.89530	.89939	.90350
.72	.90764	.91181	.91600	.92022	.92446
.73	.92873	.93302	.93734	.94169	.94607
.74	.95048	.95491	.95938	.96387	.96840
.75	.97296	.97754	.98216	.98681	.99150
.76	.99622	1.00097	1.00575	1.01058	1.01543
.77	1.02033	1.02526	1.03023	1.03524	1.04028
.78	1.04537	1.05050	1.05567	1.06088	1.06613
.79	1.07143	1.07677	1.08216	1.08760	1.09308
.80	1.09861	1.10419	1.10982	1.11551	1.12124
.81	1.12703	1.13287	1.13877	1.14473	1.15074
.82	1.15682	1.16295	1.16915	1.17541	1.18174
.83	1.18814	1.19460	1.20113	1.20774	1.21442
.84	1.22117	1.22801	1.23492	1.24191	1.24899
.85	1.25615	1.26340	1.27075	1.27818	1.28571

$r = 0.860(0.001)0.999$

r	0	1	2	3	4	5	6	7	8	9
.86	1.29334	1.29720	1.30108	1.30498	1.30891	1.31287	1.31686	1.32087	1.32491	1.32898
.87	1.33308	1.33721	1.34137	1.34555	1.34977	1.35403	1.35831	1.36262	1.36697	1.37135
.88	1.37577	1.38022	1.38470	1.38922	1.39378	1.39838	1.40301	1.40768	1.41239	1.41714
.89	1.42193	1.42676	1.43163	1.43654	1.44150	1.44651	1.45156	1.45665	1.46179	1.46698
.90	1.47222	1.47751	1.48285	1.48824	1.49368	1.49918	1.50473	1.51034	1.51601	1.52174
.91	1.52752	1.53337	1.53928	1.54526	1.55130	1.55741	1.56359	1.56984	1.57616	1.58256
.92	1.58903	1.59558	1.60221	1.60892	1.61571	1.62260	1.62957	1.63663	1.64379	1.65104
.93	1.65839	1.66584	1.67340	1.68107	1.68885	1.69674	1.70475	1.71288	1.72114	1.72953
.94	1.73805	1.74671	1.75552	1.76447	1.77358	1.78284	1.79227	1.80188	1.81166	1.82162
.95	1.83178	1.84214	1.85270	1.86349	1.87450	1.88574	1.89723	1.90898	1.92100	1.93331
.96	1.94591	1.95882	1.97207	1.98566	1.99961	2.01395	2.02870	2.04388	2.05952	2.07565
.97	2.09230	2.10950	2.12730	2.14574	2.16486	2.18472	2.20539	2.22692	2.24940	2.27291
.98	2.29756	2.32346	2.35074	2.37958	2.41014	2.44266	2.47741	2.51472	2.55499	2.59875
.99	2.64665	2.69958	2.75873	2.82574	2.90307	2.99448	3.10630	3.25039	3.45338	3.80020

9. ORDER STATISTICS

9.1. Expected Values of Order Statistics

a. Introduction

Consider a sample $(x_1, x_2, \ldots, x_n)$ of size n from a standard normal distribution. Let these observations be arranged in increasing order of magnitude as follows

$$x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}.$$

Table 9.1 provides the expected value of $x_{(i)}$ given by

$$Ex_{(i)} = \int_{-\infty}^{\infty} \frac{n!}{(i-1)!(n-i)} x\left[\int_{-\infty}^{x} N(w)dw\right]^{i-1} \left[\int_{x}^{\infty} N(w)dw\right]^{n-i} N(x)dx$$

for $i = [(n+1)/2]$ (1) n, $n = 2(1)30$. For $i < [(n+1)/2]$ the expected values are obtained using the relation

$$Ex_{(i)} = -Ex_{(n-i+1)}.$$

b. Applications

Table 9.1 is useful in the analysis of ordinal data where one has to replace the ranks by the expected values of the corresponding normal order statistics. Here the next step often involves an analysis of variance of these assigned scores. The sums of squares of the expected values given in Table 9.1 are useful in these calculations. See also the explanatory notes preceding Table 10.3 in this connection.

Another use of Table 9.1 is in obtaining factors by which the range or a quasi-range, in a sample of size n from the normal population $N(\mu, \sigma)$, has to be multiplied to give an estimate of the standard deviation σ. Thus we see from Table 9.1 that in a sample of size 20, $[x_{(18)} - x_{(3)}] \div 2.26$ provides an unbiased estimate of the population standard deviation, since for $n = 20$, $Ex_{(18)} = Ex_{(n-2)} = 1.13\sigma$ and $Ex_{(3)} = -Ex_{(n-3+1)} = -1.13\sigma$.

c. Another table of expected values

Harter, H. L. (1960): *Expected values of Normal Order Statistics*, Technical report 60–292, Aeronautical Research Laboratories, Wright-Patterson Air Force Base, June 1960.

Expected values to five places of decimal for $n = 2(1)100$ and for selected values upto $n = 400$.

TABLE 9.1. EXPECTED VALUES OF ORDER STATISTICS $x_{(i)}$ IN SAMPLES FROM A STANDARD NORMAL

order	$n=$	2	3	4	5	6	7	8	9	10
n		.56	.85	1.03	1.16	1.27	1.35	1.42	1.49	1.54
$n-1$			0	.30	.50	.64	.76	.85	.93	1.00
$n-2$					0	.20	.35	.47	.57	.66
$n-3$							0	.15	.27	.38
$n-4$									0	.12
$\sum_{i=1}^{n} Ex_{(i)}^2$		0.6272	1.4450	2.3018	3.1912	4.1250	5.0452	5.9646	6.9656	7.9320

TABLE 9.1 (*continued*). EXPECTED VALUES OF ORDER STATISTICS $x_{(i)}$ IN SAMPLES FROM A STANDARD NORMAL DISTRIBUTION

order	$n=11$	12	13	14	15	16	17	18	19	20
n	1.59	1.63	1.67	1.70	1.74	1.76	1.79	1.82	1.84	1.87
$n-1$	1.06	1.12	1.16	1.21	1.25	1.28	1.32	1.35	1.38	1.41
$n-2$	.73	.79	.85	.90	.95	.99	1.03	1.07	1.10	1.13
$n-3$	.46	.54	.60	.66	.71	.76	.81	.85	.89	92
$n-4$	.22	.31	.39	.46	.52	.57	.62	.67	.71	75
$n-5$	0	.10	.19	.27	.34	.39	.45	.50	.55	.59
$n-6$			0	.09	.17	.23	.30	.35	.40	.45
$n-7$					0	.08	.15	.21	.26	.31
$n-8$							0	.07	.13	.19
$n-9$									0	.06
$\sum_{i=1}^{n} Ex_{(i)}^2$	8.8892	9.8662	10.8104	11.7846	12.8232	13.6600	14.7258	15.7454	16.6864	17.7144

order	$n=21$	22	23	24	25	26	27	28	29	30
n	1.89	1.91	1.93	1.95	1.97	1.98	2.00	2.01	2.03	2.04
$n-1$	1.43	1.46	1.48	1.50	1.52	1.54	1.56	1.58	1.60	1.62
$n-2$	1.16	1.19	1.21	1.24	1.26	1.29	1.31	1.33	1.35	1.36
$n-3$	.95	.98	1.01	1.04	1.07	1.09	1.11	1.14	1.16	1.18
$n-4$	.78	.82	.85	.88	.91	.93	.96	.98	1.00	1.03
$n-5$	.63	.67	.70	.73	.76	.79	.82	.85	.87	.89
$n-6$	.49	.53	.57	.60	.64	.67	.70	.73	.75	.78
$n-7$	.36	.41	.45	.48	.52	.55	.58	.61	.64	.67
$n-8$	.24	.29	.33	.37	.41	.44	.48	.51	.54	.57
$n-9$	.12	.17	.22	.26	.30	.34	.38	.41	.44	.47
$n-10$	0	.06	.11	.16	.20	.24	.28	.32	.35	.38
$n-11$			0	.05	.10	.14	.19	.22	.26	.29
$n-12$					0	.05	.09	.13	.17	.21
$n-13$							0	.04	.09	.12
$n-14$									0	.04
$\sum_{i=1}^{n} Ex_{(i)}^2$	8.6242	19 6862	20.6176	21.6040	22.6352	23.5470	24.5992	25.5808	26.5806	27.5454

order	$n=31$	32	33	34	35	36	37	38	39	40
n	2.06	2.07	2.08	2.09	2.11	2.12	2.13	2.14	2.15	2.16
$n-1$	1.63	1.65	1.66	1.68	1.69	1.70	1.72	1.73	1.74	1.75
$n-2$	1.38	1.40	1.42	1.43	1.45	1.46	1.48	1.49	1.50	1.52
$n-3$	1.20	1.22	1.23	1.25	1.27	1.28	1.30	1.32	1.33	1.34
$n-4$	1.05	1.07	1.09	1.11	1.12	1.14	1.16	1.17	1.19	1.20
$n-5$	.92	.94	.96	.98	1.00	1.02	1.03	1.05	1.07	1.08
$n-6$	.80	.82	.85	.87	.89	.91	.92	.94	.96	.98
$n-7$	.69	.72	.74	.76	.79	.81	.83	.85	.86	.88
$n-8$	.60	.62	.65	.67	.69	.72	.73	.75	.77	.79
$n-9$	.50	.53	.56	.58	.60	.63	.65	.67	.69	.71
$n-10$	.41	.44	.47	.50	.52	.54	.57	.59	.61	.63
$n-11$	.33	.36	.39	.41	.44	.47	.49	.51	.54	.56
$n-12$	.24	.28	.31	.34	.36	.39	.42	.44	.46	.49
$n-13$	.16	.20	.23	.26	.29	.32	.34	.37	.39	.42
$n-14$	.08	.12	.15	.18	.22	.24	.27	.30	.33	.35
$n-15$	0	.04	.08	.11	.14	.17	.20	.23	.26	.28
$n-16$			0	.04	.07	.10	.14	.16	.19	.22
$n-17$					0	.03	.07	.10	.13	.16
$n-18$							0	.03	.06	.09
$n-19$									0	.03
$\sum_{i=1}^{n} Ex_{(i)}^2$	28.5730	29.5960	30.5562	31.5152	32.5618	33.5166	34.5346	35.4840	36.4414	37.4288

9.2. Fractiles of a Normal Distribution

a. Fractile mean and variance

For a standard normal distribution with the density function $N(x) = (2\pi)^{-\frac{1}{2}} e^{-x^2/2}$ $(-\infty < x < \infty)$, consider the system of intervals $(a_i, a_{i+1}]$ $i = 1, 2, \ldots, g$ where $a_1 = -\infty$, $a_{g+1} = \infty$ and the $g-1$ other a's are chosen such that

$$\int_{a_i}^{a_{i+1}} N(x)dx = \frac{1}{g} \quad (i = 1, 2, \ldots, g).$$

The interval $(a_i, a_{i+1}]$ will be referred to as the i-th g-fractile interval of the standard normal distribution. Table 9.2 gives the mean

$$\mu_{[i, g]} = g \int_{a_i}^{a_{i+1}} xN(x)dx$$

and variance.

$$\sigma^2_{[i, g]} = g \int_{a_i}^{a_{i+1}} x^2N(x)dx - \mu^2_{[i, g]}$$

in the i-th g-fractile interval for $i = 1(1)g$, $g = 2(1)20$

b. Application : graphical tests of normality

Let $x_1, x_2, \ldots, x_n$ be a sample of n observations from a population. Two graphical tests are described for examining whether the parent population is normal.

(i) *Normal probability graph*

Denote the ordered observations by $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Consider the pairs $(d_1, x_{(1)}), \ldots, (d_{n-1}, x_{(n-1)})$ where d_i is the standard normal deviate corresponding to the cumulative probability of i/n. The values of d_i can be obtained from Table 3.1, by inverse interpolation if necessary. Then the $(d_i, x_{(i)})$, $i = 1, 2, \ldots, (n-1)$ are plotted on a graph paper with orthogonal axes (x and y) with d_i on x-axis and $x_{(i)}$ on y-axis. If the parent population is normal the points will lie close to a straight line.

(ii) *Fractile graph**

We consider the order observations $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ as in method I. Now divide the observations into a chosen number, g, of groups such that each group consists of $h = n/g$ consecutive order observations. The groups so obtained are called fractile groups. The i-th fractile group consists of the observations

$$x_{(ih)}, x_{(ih+1)}, \ldots, x_{(ih+h-1)}$$

The sample i-th fractile mean is the average of the observations in the i-th fractile group and is represented by

$$\bar{x}_{[i, g]} = \frac{x_{(ih)} + \ldots + x_{(ih+h-1)}}{h}$$

* The fractile graphical analysis was recently developed by Mahalanobis (Econometrica, 28, 325 351). It is capable of a very wide application. The particular application of testing for normality was suggested by A. Linder in the convocation address at the Indian Statistical Institute in 1963.

We consider the pairs

$$(\mu_{[i,g]}, \bar{x}_{[i,g]}), \quad i = 1, 2, \ldots, g$$

where $\mu_{[i,g]}$ are the fractile means of the population as defined in section 1, and tabulated in Table 9.2. Then the g points $(\mu_{[i,g]}, \bar{x}_{[i,g]})$, $i = 1, 2, \ldots, g$ are plotted on a two dimensional chart representing $\mu_{[i,g]}$ on x-axis and $\bar{x}_{[i,g]}$ on y-axis. If the parent population is normal the graph will be close to a straight line.

Example : Given 100 independent observations on log weight of an individual, it is required to examine whether the distribution of log weight is normal.

first half sample

2.081	2.204	2.130	2.207	2.111	2.189	2.230	2.150	2.208	2.191
2.094	2.174	2.177	2.170	2.098	2.105	2.198	2.085	2.145	2.131
2.120	2.186	2.097	2.171	2.168	2.215	2.096	2.116	2.132	2.062
2.112	2.078	2.171	2.177	2.151	2.241	2.167	2.105	2.175	2.151
2.103	2.144	2.204	2.189	2.108	2.267	2.173	2.076	2.283	2.165

second half sample

2.168	2.046	2.192	2.258	2.236	2.098	2.210	2.267	2.137	2.179
2.159	2.125	2.127	2.138	2.102	2.166	2.192	2.212	2.143	2.171
2.185	2.236	2.075	2.079	2.162	2.052	2.153	2.206	2.235	2.215
2.239	2.046	2.131	2.152	2.116	2.172	2.272	2.086	2.124	2.139
2.134	2.140	2.115	2.122	2.132	2.197	2.137	2.143	2.124	2.135

We illustrate the fractile graph method which is less well-known than the probability graph method.

In such problems involving graphical analysis of data, it is useful to split the sample into two independent half samples (of 50 observations each in the present case) and draw the fractile graph for each half sample and also for the combined sample. Such a procedure would enable us to examine the consistency between parallel samples and also to have an idea of the magnitude of the sampling error (separation between half sample graphs) involved. The observed deviation from a straight line of the fractile graph for the combined sample has to be judged against sampling error, i.e., the deviation to be expected due to sampling.

fractile group	fractile mean			
	half sample 1	half sample 2	combined sample	theoretical (from Table 9.2)
1	2.076	2.060	2.068	−1.755
2	2.096	2.102	2.097	−1.045
3	2.108	2.125	2.117	−0.677
4	2.126	2.134	2.131	−0.387
5	2.150	2.139	2.143	−0.126
6	2.169	2.154	2.162	0.126
7	2.175	2.171	2.174	0.387
8	2.186	2.194	2.190	0.677
9	2.204	2.222	2.211	1.045
10	2.247	2.254	2.253	1.755

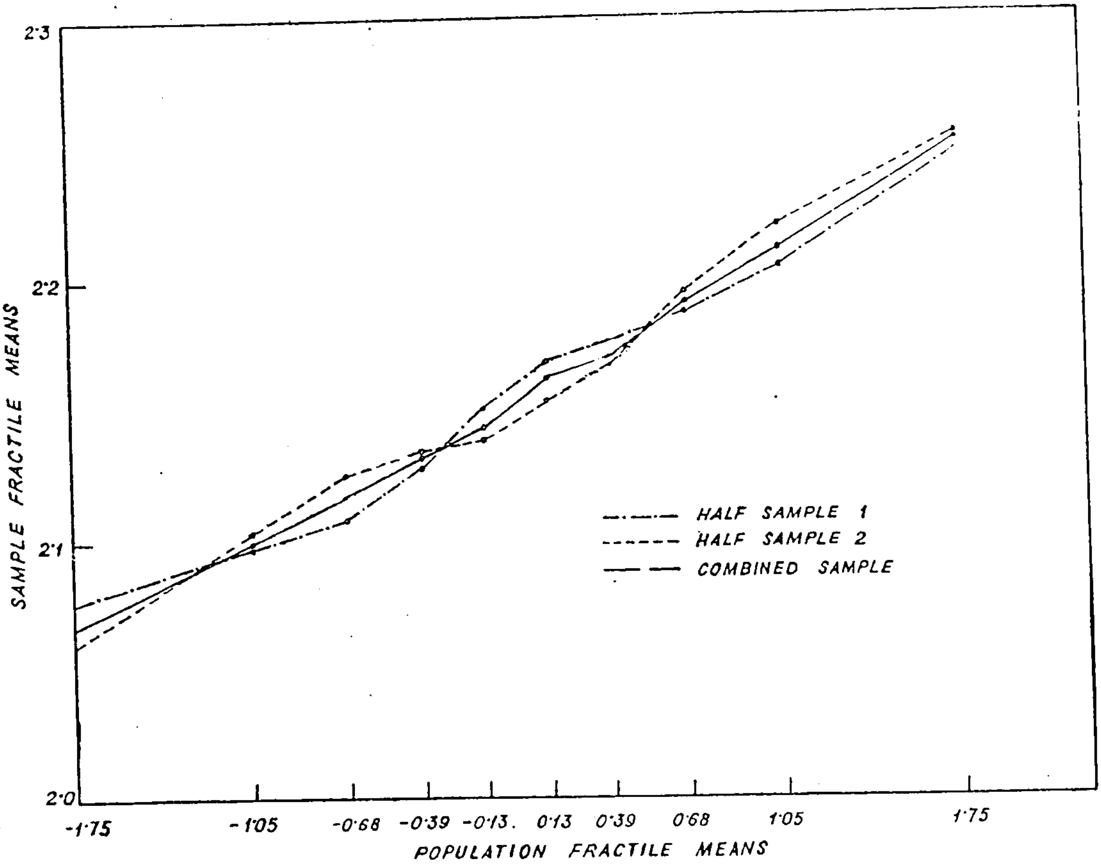
The two fractile graphs based on samples of 100 observations are in the chart on page 94. The deviations from a straight line appear to be small compared to the difference between the half sample fractile graphs.

TABLE 9.2. MEAN AND VARIANCE FOR FRACTILES OF A STANDARD NORMAL DISTRIBUTION

For each combination of a value of g and a fractile number there are two entries, of which the top entry represents the mean and the lower entry, the variance

					fractiles					
g	1	2	3	4	5	6	7	8	9	10
2	−0.7979	0.7979								
	0.3634	0.3634								
3	−1.0908	0	1.0908							
	0.2800	0.0603	0.2800							
4	−1.2711	−0.3247	0.3247	1.2711						
	0.2416	0.0372	0.0372	0.2416						
5	−1.3998	−0.5319	0	0.5319	1.3998					
	0.2186	0.0284	0.0212	0.0284	0.2186					
6	−1.4991	−0.6825	−0.2121	−0.2121	0.6825	1.4991				
	0.2029	0.0236	0.0154	0.0154	0.0236	0.2029				
7	−1.5795	−0.7998	−0.3684	0	0.3684	0.7998	1.5795			
	0.1914	0.0206	0.0123	0.0108	0.0123	0.0206	0.1914			
8	−1.6468	−0.8954	−0.4913	−0.1580	0.1580	0.4913	0.8954	1.6468		
	0.1824	0.0186	0.0105	0.0084	0.0084	0.0105	0.0186	0.1824		
9	−1.7046	−0.9757	−0.5922	−0.2832	0	0.2832	0.5922	0.9757	1.7046	
	0.1751	0.0170	0.0092	0.0070	0.0065	0.0070	0.0092	0.0170	0.1751	
10	−1.7550	−1.0446	−0.6773	−0.3865	−0.1260	0.1260	0.3865	0.6773	1.0446	1.7550
	0.1691	0.0159	0.0083	0.0061	0.0053	0.0053	0.0061	0.0083	0.0159	0.1691
11	−1.7997	−1.1050	−0.7507	−0.4741	−0.2304	0*				
	0.1640	0.0149	0.0077	0.0054	0.0046	0.0043				
12	−1.8398	−1.1585	−0.8151	−0.5499	−0.3193	−0.1048*				
	0.1597	0.0141	0.0071	0.0049	0.0040	0.0037				
13	−1.8760	−1.2064	−0.8723	−0.6165	−0.3964	−0.1943	0*			
	0.1559	0.0135	0.0067	0.0045	0.0036	0.0032	0.0031			
14	−1.9092	−1.2499	−0.9237	−0.6759	−0.4645	−0.2723	−0.0898*			
	0.1525	0.0129	0.0063	0.0042	0.0033	0.0029	0.0027			
15	−1.9396	−1.2895	−0.9703	−0.7294	−0.5252	−0.3411	−0.1681	0*		
	0.1495	0.0125	0.0060	0.0040	0.0031	0.0026	0.0024	0.0023		
16	−1.9677	−1.3259	−1.0125	−0.7779	−0.5800	−0.4027	−0.2375	−0.0785*		
	0.1467	0.0121	0.0057	0.0038	0.0029	0.0024	0.0022	0.0021		
17	−1.9939	−1.3596	−1.0520	−0.8223	−0.6298	−0.4584	−0.2996	−0.1481	0*	
	0.1443	0.0117	0.0055	0.0036	0.0027	0.0022	0.0020	0.0019	0.0018	
18	−2.0183	−1.3908	−1.0882	−0.8631	−0.6754	−0.5090	−0.3558	−0.2106	−0.0697*	
	0.1420	0.0114	0.0053	0.0034	0.0026	0.0021	0.0018	0.0017	0.0016	
19	−2.0412	−1.4200	−1.1218	−0.9009	−0.7174	−0.5565	−0.4071	−0.2672	−0.1324	0*
	0.1400	0.0111	0.0051	0.0033	0.0024	0.0020	0.0017	0.0016	0.0015	0.0015
20	−2.0627	−1.4473	−1.1532	−0.9361	−0.7563	−0.5983	−0.4541	−0.3189	−0.1892	0.0627*
	0.1380	0.0108	0.0050	0.0032	0.0023	0.0019	0.0016	0.0015	0.0014	0.0013

*For $g \geqslant 11$, mean and variance are given only for the first $g/2$ fractiles if g is even and $(g+1)/2$ fractiles if g is odd. The rest of the fractile means and variances for any g can be written down by symmetry, the sign being changed in the case of the means. Thus the 7th, 8th,............fractile means for $g=11$ are 0.2304, 0.4741, ...etc., and variances 0.0046, 0.0054, ...etc. For $g=12$, the 7th fractile mean and variance are 0.1048 and 0.0037 and so on.



9.3. The Maximum Observation

Table 9.3 provides the upper 5%, 1% and 0.1% points of the maximum observation $x_{(n)}$ in a sample of size n from $N(0, 1)$ for $n = 1\ (1)30$. Owing to the symmetry of $N(0, 1)$ the same table is also applicable to $-x_{(1)}$, $x_{(1)}$ being the minimum observation in a sample of size n.

It is known from experience that the average and standard deviation of the weight of individual cigarettes are 6.00 and 1.50 units. 5 cigarettes selected at random weighed 6.00, 9.50, 4.41, 7.51 and 4.29 units. Examine if the maximum observation is an outlier The extreme standardised deviate $(9.50-6.00)/1.50 = 2.33$. This exceeds 2.319 the upper 5% value given in Table 9.3 for $n = 5$. Hence one has reasons to suspect the maximum observation.

TABLE 9.3. UPPER PERCENTAGE POINTS OF THE MAXIMUM OBSERVATION

n	0.1%	1%	5%	n	0.1%	1%	5%	n	0.1%	1%	5%
1	3.090	2.326	1.645	11	3.743	3.117	2.601	21	3.902	3.303	2.815
2	3.290	2.575	1.955	12	3.765	3.143	2.630	22	3.914	3.316	2.830
3	3.403	2.712	2.121	13	3.785	3.166	2.657	23	3.924	3.328	2.844
4	3.481	2.806	2.234	14	3.803	3.187	2.682	24	3.934	3.340	2.857
5	3.540	2.877	2.319	15	3.820	3.207	2.705	25	3.944	3.351	2.870
6	3.588	2.934	2.386	16	3.836	3.226	2.726	26	3.954	3.362	2.883
7	3.628	2.981	2.442	17	3.851	3.243	2.746	27	3.963	3.373	2.895
8	3.662	3.022	2.490	18	3.865	3.259	2.765	28	3.971	3.383	2.906
9	3.692	3.057	2.531	19	3.878	3.275	2.783	29	3.980	3.392	2.917
10	3.719	3.089	2.568	20	3.890	3.289	2.799	30	3.988	3.402	2.928

9.4. The Extreme Studentised Deviate from the Sample Mean

Table 9.4 gives the upper 1% and 5% points of $\frac{x_{(n)}-\bar{x}}{s_\nu}\left(\text{or } \frac{\bar{x}-x_{(1)}}{s_\nu}\right)$ computed from a sample of size n drawn from $N(\mu, \sigma)$, where $\bar{x}$ is the sample mean $x_{(1)}$ and $x_{(n)}$ are the minimum and the maximum observation in the sample and s_ν^2 is an independent unbiased estimate for σ^2 based on ν degree of freedom.

Table 9.4 is useful in deciding whether to reject an allegedly outlying observation, as in 9.3 when the population mean and variance are unknown.

TABLE 9.4. UPPER PERCENTAGE POINTS OF THE EXTREME STUDENTISED DEVIATE FROM THE SAMPLE MEAN

	1%								5%							
ν \ n	3	4	5	6	7	8	9	12	3	4	5	6	7	8	9	12
10	2.78	3.10	3.32	3.48	3.62	3.73	3.82	4.04	2.01	2.27	2.46	2.60	2.72	2.81	2.89	3.08
11	2.72	3.02	3.24	3.39	3.52	3.63	3.72	3.93	1.98	2.24	2.42	2.56	2.67	2.76	2.84	3.03
12	2.67	2.96	3.17	3.32	3.45	3.55	3.64	3.84	1.96	2.21	2.39	2.52	2.63	2.72	2.80	2.98
13	2.63	2.92	3.12	3.27	3.38	3.48	3.57	3.76	1.94	2.19	2.36	2.50	2.60	2.69	2.76	2.94
14	2.60	2.88	3.07	3.22	3.33	3.43	3.51	3.70	1.93	2.17	2.34	2.47	2.57	2.66	2.74	2.91
15	2.57	2.84	3.03	3.17	3.29	3.38	3.46	3.65	1.91	2.15	2.32	2.45	2.55	2.64	2.71	2.88
16	2.54	2.81	3.00	3.14	3.25	3.34	3.42	3.60	1.90	2.14	2.31	2.43	2.53	2.62	2.69	2.86
17	2.52	2.79	2.97	3.11	3.22	3.31	3.38	3.56	1.89	2.13	2.29	2.42	2.52	2.60	2.67	2.84
18	2.50	2.77	2.95	3.08	3.19	3.28	3.35	3.53	1.88	2.11	2.28	2.40	2.50	2.58	2.65	2.82
19	2.49	2.75	2.93	3.06	3.16	3.25	3.33	3.50	1.87	2.11	2.27	2.39	2.49	2.57	2.64	2.80
20	2.47	2.73	2.91	3.04	3.14	3.23	3.30	3.47	1.87	2.10	2.26	2.38	2.47	2.56	2.63	2.78
24	2.42	2.68	2.84	2.97	3.07	3.16	3.23	3.38	1.84	2.07	2.23	2.34	2.44	2.52	2.58	2.74
30	2.38	2.62	2.79	2.91	3.01	3.08	3.15	3.30	1.82	2.04	2.20	2.31	2.40	2.48	2.54	2.69
40	2.34	2.57	2.73	2.85	2.94	3.02	3.08	3.22	1.80	2.02	2.17	2.28	2.37	2.44	2.50	2.65
60	2.29	2.52	2.68	2.79	2.88	2.95	3.01	3.15	1.78	1.99	2.14	2.25	2.33	2.41	2.47	2.61
120	2.25	2.48	2.62	2.73	2.82	2.89	2.95	3.08	1.76	1.96	2.11	2.22	2.30	2.37	2.43	2.57
∞	2.22	2.43	2.57	2.68	2.76	2.83	2.88	3.01	1.74	1.94	2.08	2.18	2.27	2.33	2.39	2.52

9.5. W TEST FOR NORMALITY

a. Introduction

Of all the known tests for normality, the W test given by Shapiro and Wilk (*Biometrika* **52**, 1965) is generally efficient against a wide spectrum of non-normal alternatives, and can be effective even when the sample size is small. Given n observations $x_1, x_2, \ldots, x_n$, the W statistic is computed as follows :

(i) Rearrange the observations to obtain the ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$.

(ii) Compute $\bar{x} = (\Sigma x)/n$ and $S^2 = \Sigma x_i^2 - n\bar{x}^2$.

(iii) Compute

$$b = \sum_{i=1}^{k} a_{n-i+1}[x_{(n-i+1)} - x_{(i)}]$$

where $k = n/2$ if n is even and $k = (n-1)/2$ if n is odd. The values of a_{n-i+1} are given in Table 9.5 for $n = 3(1)50$ and $i = 1$ to $\frac{n}{2}\left(\text{ or } \frac{n-1}{2}\right)$

(iv) Then compute $W = b^2/S^2$.

The hypothesis of normality is rejected at $p\%$ level if $W \leqslant W_p$. The critical values of W_p are given in Table 9.6 for $p = 1, 2, 5, 10, 50$ and $n = 3(1)50$. (Note that the exact distribution of W is not known and the percentage points are obtained by simulation and appropriate smoothing).

b. Example

Ten observations on weights of cigarettes (in coded units) after ordering are as follows :

$$x_{(1)} = 303,\ x_{(2)} = 338,\ x_{(3)} = 406,\ x_{(4)} = 457,\ x_{(5)} = 461$$

$$x_{(6)} = 469,\ x_{(7)} = 474,\ x_{(8)} = 489,\ x_{(9)} = 515,\ x_{(10)} = 583$$

$$\bar{x} = 449.5, \quad S^2 = 60628.$$

The value of $k = 5$, since $n = 10$. From table 9.5 we have

$$a_{10} = 0.5739,\ a_9 = 0.3291,\ \ldots,\ a_6 = 0.0399$$

$$\begin{aligned} b &= a_{10}(x_{(10)} - x_{(1)}) + a_9(x_{(9)} - x_{(2)}) + \ldots + a_6(x_{(6)} - x_{(5)}) \\ &= 0.5739(583-303) + 0.3291(515\text{-}338) + \ldots + 0.0399(469\text{-}461) \\ &= 239.113. \end{aligned}$$

$$W = \frac{(239.113)^2}{60628} = 0.943.$$

and the 5% critical value of $W = 0.842$ from Table 9.6 for $n = 10$. Hence on the basis of limited available data, there is no reason to reject the hypothesis of normality.

TABLE 9.5. COEFFICIENTS (a_{n-i+1}) USED IN W TEST FOR NORMALITY, FOR $n = 3(1)50$

$i \backslash n$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0.7071	0.6872	0.6646	0.6431	0.6233	0.6052	0.5888	0.5739	0.5601	0.5475	0.5359	0.5251	0.5150	0.5056	0.4968	0.4886
2		0.1677	0.2413	0.2806	0.3031	0.3164	0.3244	0.3291	0.3315	0.3325	0.3325	0.3318	0.3306	0.3290	0.3273	0.3253
3				0.0875	0.1401	0.1743	0.1976	0.2141	0.2260	0.2347	0.2412	0.2460	0.2495	0.2521	0.2540	0.2553
4						0.0561	0.0947	0.1224	0.1429	0.1586	0.1707	0.1802	0.1878	0.1939	0.1988	0.2027
5								0.0399	0.0695	0.0922	0.1099	0.1240	0.1353	0.1447	0.1524	0.1587
6										0.0303	0.0539	0.0727	0.0880	0.1005	0.1109	0.1197
7												0.0240	0.0433	0.0593	0.0725	0.0837
8														0.0196	0.0359	0.0496
9																0.0163

$i \backslash n$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
1	0.4808	0.4734	0.4643	0.4590	0.4542	0.4493	0.4450	0.4407	0.4366	0.4328	0.4291	0.4254	0.4200	0.4188	0.4156	0.4127
2	0.3232	0.3211	0.3185	0.3156	0.3126	0.3098	0.3069	0.3043	0.3018	0.2992	0.2968	0.2944	0.2921	0.2898	0.2876	0.2854
3	0.2561	0.2565	0.2578	0.2571	0.2563	0.2554	0.2543	0.2533	0.2522	0.2510	0.2499	0.2487	0.2475	0.2463	0.2451	0.2439
4	0.2059	0.2085	0.2119	0.2131	0.2139	0.2145	0.2148	0.2151	0.2152	0.2151	0.2150	0.2148	0.2145	0.2141	0.2137	0.2132
5	0.1641	0.1686	0.1736	0.1764	0.1787	0.1807	0.1822	0.1836	0.1848	0.1857	0.1864	0.1870	0.1874	0.1878	0.1880	0.1882
6	0.1271	0.1334	0.1399	0.1443	0.1480	0.1512	0.1539	0.1563	0.1584	0.1601	0.1616	0.1630	0.1641	0.1651	0.1660	0.1667
7	0.0932	0.1013	0.1092	0.1150	0.1201	0.1245	0.1283	0.1316	0.1346	0.1372	0.1395	0.1415	0.1433	0.1449	0.1463	0.1475
8	0.0612	0.0711	0.0804	0.0878	0.0941	0.0997	0.1046	0.1089	0.1128	0.1162	0.1192	0.1219	0.1243	0.1265	0.1284	0.1301
9	0.0303	0.0422	0.0530	0.0618	0.0696	0.0764	0.0823	0.0876	0.0923	0.0965	0.1002	0.1036	0.1066	0.1093	0.1118	0.1140
10		0.0140	0.0263	0.0368	0.0459	0.0539	0.0610	0.0672	0.0728	0.0778	0.0822	0.0862	0.0899	0.0931	0.0961	0.0988
11				0.0122	0.0228	0.0321	0.0403	0.0476	0.0540	0.0598	0.0650	0.0697	0.0739	0.0777	0.0812	0.0844
12						0.0107	0.0200	0.0284	0.0358	0.0424	0.0483	0.0537	0.0585	0.0629	0.0669	0.0706
13								0.0094	0.0178	0.0253	0.0320	0.0381	0.0435	0.0485	0.0530	0.0572
14										0.0084	0.0159	0.0227	0.0289	0.0344	0.0395	0.0441
15												0.0076	0.0144	0.0206	0.0262	0.0314
16														0.0068	0.0131	0.0187
17																0.0062

$i \backslash n$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
1	0.4096	0.4068	0.4040	0.4015	0.3989	0.3964	0.3940	0.3917	0.3894	0.3872	0.3850	0.3830	0.3808	0.3789	0.3770	0.3751
2	0.2834	0.2813	0.2794	0.2774	0.2755	0.2737	0.2719	0.2701	0.2684	0.2667	0.2651	0.2635	0.2620	0.2604	0.2589	0.2574
3	0.2427	0.2415	0.2403	0.2391	0.2380	0.2368	0.2357	0.2345	0.2334	0.2323	0.2313	0.2302	0.2291	0.2281	0.2271	0.2260
4	0.2127	0.2121	0.2116	0.2110	0.2104	0.2098	0.2091	0.2085	0.2078	0.2072	0.2065	0.2058	0.2052	0.2045	0.2038	0.2032
5	0.1883	0.1883	0.1883	0.1881	0.1880	0.1878	0.1876	0.1874	0.1871	0.1868	0.1865	0.1862	0.1859	0.1855	0.1851	0.1847
6	0.1673	0.1678	0.1683	0.1686	0.1689	0.1691	0.1693	0.1694	0.1695	0.1695	0.1695	0.1695	0.1695	0.1693	0.1692	0.1691
7	0.1487	0.1496	0.1505	0.1513	0.1520	0.1526	0.1531	0.1535	0.1539	0.1542	0.1545	0.1548	0.1550	0.1551	0.1553	0.1554
8	0.1317	0.1331	0.1344	0.1356	0.1366	0.1376	0.1384	0.1392	0.1398	0.1405	0.1410	0.1415	0.1420	0.1423	0.1427	0.1430
9	0.1160	0.1179	0.1196	0.1211	0.1225	0.1237	0.1249	0.1259	0.1269	0.1278	0.1286	0.1293	0.1300	0.1306	0.1312	0.1317
10	0.1013	0.1036	0.1056	0.1075	0.1092	0.1108	0.1123	0.1136	0.1149	0.1160	0.1170	0.1180	0.1189	0.1197	0.1205	0.1212
11	0.0873	0.0900	0.0924	0.0947	0.0967	0.0986	0.1004	0.1020	0.1035	0.1049	0.1062	0.1073	0.1085	0.1095	0.1105	0.1113
12	0.0739	0.0770	0.0798	0.0824	0.0848	0.0870	0.0891	0.0909	0.0927	0.0943	0.0959	0.0972	0.0986	0.0998	0.1010	0.1020
13	0.0610	0.0645	0.0677	0.0706	0.0733	0.0759	0.0782	0.0804	0.0824	0.0842	0.0860	0.0876	0.0892	0.0906	0.0919	0.0932
14	0.0484	0.0523	0.0559	0.0592	0.0622	0.0651	0.0677	0.0701	0.0724	0.0745	0.0765	0.0783	0.0801	0.0817	0.0832	0.0846
15	0.0361	0.0404	0.0444	0.0481	0.0515	0.0546	0.0575	0.0602	0.0628	0.0651	0.0673	0.0694	0.0713	0.0731	0.0748	0.0764
16	0.0239	0.0287	0.0331	0.0372	0.0409	0.0444	0.0476	0.0506	0.0534	0.0560	0.0584	0.0607	0.0628	0.0648	0.0667	0.0685
17	0.0119	0.0172	0.0220	0.0264	0.0305	0.0343	0.0379	0.0411	0.0442	0.0471	0.0497	0.0522	0.0546	0.0568	0.0588	0.0608
18		0.0057	0.0110	0.0158	0.0203	0.0244	0.0283	0.0318	0.0352	0.0383	0.0412	0.0439	0.0465	0.0489	0.0511	0.0532
19				0.0053	0.0101	0.0146	0.0188	0.0227	0.0263	0.0296	0.0328	0.0357	0.0385	0.0411	0.0436	0.0459
20						0.0049	0.0094	0.0136	0.0175	0.0211	0.0245	0.0277	0.0307	0.0335	0.0361	0.0386
21								0.0045	0.0087	0.0126	0.0163	0.0197	0.0229	0.0259	0.0288	0.0314
22										0.0042	0.0081	0.0118	0.0153	0.0185	0.0215	0.0244
23												0.0039	0.0076	0.0111	0.0143	0.0174
24														0.0037	0.0071	0.0104
25																0.0035

TABLE 9.6. PERCENTAGE POINTS OF W TEST FOR NORMALITY FOR $n = 3(1)50$

n	1%	2%	5%	10%	50%
3	0.753	0.756	0.767	0.789	0.959
4	0.687	0.707	0.748	0.792	0.935
5	0.686	0.715	0.762	0.806	0.927
6	0.713	0.743	0.788	0.826	0.927
7	0.730	0.760	0.803	0.838	0.928
8	0.749	0.778	0.818	0.851	0.932
9	0.764	0.791	0.829	0.859	0.935
10	0.781	0.806	0.842	0.869	0.938
11	0.792	0.817	0.850	0.876	0.940
12	0.805	0.828	0.859	0.883	0.943
13	0.814	0.837	0.866	0.889	0.945
14	0.825	0.846	0.874	0.895	0.947
15	0.835	0.855	0.881	0.901	0.950
16	0.844	0.863	0.887	0.906	0.952
17	0.851	0.869	0.892	0.910	0.954
18	0.858	0.874	0.897	0.914	0.956
19	0.863	0.879	0.901	0.917	0.957
20	0.868	0.884	0.905	0.920	0.959
21	0.873	0.888	0.908	0.923	0.960
22	0.878	0.892	0.911	0.926	0.961
23	0.881	0.895	0.914	0.928	0.962
24	0.884	0.898	0.916	0.930	0.963
25	0.888	0.901	0.918	0.931	0.964
26	0.891	0.904	0.920	0.933	0.965
27	0.894	0.906	0.923	0.935	0.965
28	0.896	0.908	0.924	0.936	0.966
29	0.898	0.910	0.926	0.937	0.966
30	0.900	0.912	0.927	0.939	0.967
31	0.902	0.914	0.929	0.940	0.967
32	0.904	0.915	0.930	0.941	0.968
33	0.906	0.917	0.931	0.942	0.968
34	0.908	0.919	0.933	0.943	0.969
35	0.910	0.920	0.934	0.944	0.969
36	0.912	0.922	0.935	0.945	0.970
37	0.914	0.924	0.936	0.946	0.970
38	0.916	0.925	0.938	0.947	0.971
39	0.917	0.927	0.939	0.948	0.971
40	0.919	0.928	0.940	0.949	0.972
41	0.920	0.929	0.941	0.950	0.972
42	0.922	0.930	0.942	0.951	0.972
43	0.923	0.932	0.943	0.951	0.973
44	0.924	0.933	0.944	0.952	0.973
45	0.926	0.934	0.945	0.953	0.973
46	0.927	0.935	0.945	0.953	0.974
47	0.928	0.936	0.946	0.954	0.974
48	0.929	0.937	0.947	0.954	0.974
49	0.929	0.937	0.947	0.955	0.974
50	0.930	0.938	0.947	0.955	0.974

9.6 Tests For Outliers

a. Introduction

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ denote a random sample of n observations from a normal population arranged in the ascending order of magnitude. Dixon (*Ann. Math. Stat.*. 22, 1951) has tabulated the percentage points of the distribution of the ratios of the form $\dfrac{x_{(n)}-x_{(n-j)}}{x_{(n)}-x_{(i)}}$ for testing whether $x_{(n)}$ is an outlier and of the form $\dfrac{x_{(j)}-x_{(1)}}{x_{(n-i)}-x_{(1)}}$ for testing whether $x_{(1)}$ is an outlier, for small values of i and j and $n \leqslant 30$. Table 9.7 gives the upper 5% and 1% points (or equivalently critical values corresponding to $\alpha = 0.05$ and $\alpha = 0.01$) of the following statistics.

$$r_{10} = \frac{x_{(2)}-x_{(1)}}{x_{(n)}-x_{(1)}} \quad \text{or} \quad \frac{x_{(n)}-x_{(n-1)}}{x_{(n)}-x_{(1)}} \quad \text{for} \quad n = 3(1)7$$

$$r_{11} = \frac{x_{(2)}-x_{(1)}}{x_{(n-1)}-x_{(1)}} \quad \text{or} \quad \frac{x_{(n)}-x_{(n-1)}}{x_{(n)}-x_{(2)}} \quad \text{for} \quad n = 8(1)10$$

$$r_{21} = \frac{x_{(3)}-x_{(1)}}{x_{(n-1)}-x_{(1)}} \quad \text{or} \quad \frac{x_{(n)}-x_{(n-2)}}{x_{(n)}-x_{(2)}} \quad \text{for} \quad n = 11(1)13$$

$$r_{22} = \frac{x_{(3)}-x_{(1)}}{x_{(n-2)}-x_{(1)}} \quad \text{or} \quad \frac{x_{(n)}-x_{(n-2)}}{x_{(n)}-x_{(3)}} \quad \text{for} \quad n = 14(1)25$$

b. Application

The main use of this table is to test whether $x_{(1)}$ or $x_{(n)}$ is an outlying observation. When this method is used for testing an extreme mean, the samples from which the means are computed should all have the same size. The recommended procedure is to use r_{10} for $n = 3$ to 7, r_{11} for $n = 8$ to 10, r_{21} for $n = 11$ to 13 and r_{22} for $n = 14$ to 25. For example, when $n = 8$, we calculate $r_{11} = \dfrac{x_{(2)}-x_{(1)}}{x_{(n-1)}-x_{(1)}}$ for testing a single outlier $x_{(1)}$ at the lower end or $r_{11} = \dfrac{x_{(n)}-x_{(n-1)}}{x_{(n)}-x_{(2)}}$ for testing a single large outlier $x_{(n)}$.

Example

Chemical analysis results of a certain chemical content for six samples are as follows :

0.470, 0.498, 0.505, 0.528, 0.564 and 0.600.

To test whether $x_{(6)}(= 0.600)$ is an outlier we compute

$$r_{10} = \frac{0.600-0.564}{0.600-0.470} = 0.28.$$

Since this is less than the critical value 0.560 for $\alpha = 0.05$, $x_{(6)}$ may not be judged to be different from the others.

TABLE 9.7. CRITERIA AND CRITICAL VALUES FOR TESTING AN EXTREME VALUE

Statistic	Number of observations n	Critical Values $\alpha = 0.05$	$\alpha = 0.01$
$r_{10} = \dfrac{x_{(2)} - x_{(1)}}{x_{(n)} - x_{(1)}}$	3	0.941	0.988
	4	0.765	0.889
	5	0.642	0.780
	6	0.560	0.698
	7	0.507	0.637
$r_{11} = \dfrac{x_{(2)} - x_{(1)}}{x_{(n-1)} - x_{(1)}}$	8	0.554	0.683
	9	0.512	0.635
	10	0.477	0.597
$r_{21} = \dfrac{x_{(3)} - x_{(1)}}{x_{(n-1)} - x_{(1)}}$	11	0.576	0.679
	12	0.546	0.642
	13	0.521	0.615
$r_{22} = \dfrac{x_{(3)} - x_{(1)}}{x_{(n-2)} - x_{(1)}}$	14	0.546	0.641
	15	0.525	0.616
	16	0.507	0.595
	17	0.490	0.577
	18	0.475	0.561
	19	0.462	0.547
	20	0.450	0.535
	21	0.440	0.524
	22	0.430	0.514
	23	0.421	0.505
	24	0.413	0.497
	25	0.406	0.489

9.7 Probability Plotting

a. Introduction

The technique of probability plotting provides a pictorial representation of the data as well as (*a*) an evaluation of the reasonableness of the assumed probability model, (*b*) estimates of the percentiles of the distribution and (c) estimates of unknown parameters of the underlying distribution.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be an ordered sample of size n from a population with probability density function $f(x)$ and cumulative distribution function $F(x)$. Then the expected value of $x_{(i)}$ is

$$E(x_{(i)}) = \frac{n!}{(i-1)!\,(n-i)!} \int_{-\infty}^{\infty} y[F(y)]^{i-1}[1-F(y)]^{n-i}dF(y) \qquad \ldots \quad (1)$$

For example if x is a uniform variate over the interval (0, 1), then

$$E(x_{(i)}) = \frac{i}{n+1} \text{ for } i = 1, 2, \ldots, n.$$

The expected values of ordered observations have been tabulated for many distributions (see Sarhan and Greenberg, *Contributions to Order Statistics*, John Wiley 1962). For distributions for which $E(x_{(i)})$ cannot be calculated exactly, the following approximation is frequently used.

$$E(x_{(i)}) = F^{-1}\left(\frac{i-c}{n-2c+1}\right) \qquad \ldots \quad (2)$$

where $F^{-1}[(i-c)/(n-2c+1)]$ is the value of x such that $\int_{-\infty}^{x} f(u)du = (i-c)/(n-2c+1)$.
that is, the $[(i-c)/(n-2c+1)]$-th fractile of the distribution and c is a number which depends on n and $f(x)$. The ordered observed values when plotted against their expected values would give a straight line passing through the origin with slope unity. The origin and slope of the plot will change if the variable is linearly transformed for plotting convenience, but the plot will remain a straight line.

The construction of specially scaled graph papers has obviated the need for calculating the expected values for many distributions. The graph paper is scaled in such a fashion that the ordered observations can be plotted directly against $100(i-c)/(n-2c+1)$, without the need of determining $E(x_{(i)})$. The correct value of c depends on $f(x)$ and n but $c = \frac{1}{2}$ can be used for a wide variety of distributions and sample sizes. The following steps are involved in preparing a probability plot for a given set of data.

(*i*) Obtain a probability paper designed for the distribution under examination.

(*ii*) Rank the observations from smallest to largest *i.e.*, $x_{(1)} \leqslant x_{(2)} \ldots \leqslant x_{(n)}$.

(*iii*) Plot $x_{(i)}$ against $100\ (i-\frac{1}{2})/n$ on the probability paper.

If the chosen model is correct, the points should cluster around a line, although there will be some deviations because of random sampling fluctuations. If a straight line 'appears' to fit the data, find the best fitting line using a suitable method. The probability plot for the normal distribution is discussed in Section ***b*** of 9.2; we shall briefly describe below the probability plots for the Weibull and Type I Extreme value distributions.

b. Weibull distribution

The cumulative probability distribution function for the Weibull distribution is

$$F(x) = 1-\exp\left[-\left(\frac{x}{\sigma}\right)^{\eta}\right], \; 0 \leqslant x < \infty \quad \eta > 0 \quad \sigma > 0$$

where σ and η are the scale and shape parameters respectively. We have, with logarithms taken to base e,

$$\log\log\frac{1}{1-F(x)} = \eta\log x - \eta\log\sigma. \qquad \ldots \quad (3)$$

Thus for a Weibull variate, $\log\log[1-F(x)]^{-1}$ has a straight line relationship with $\log x$. The axes of the probability paper are scaled so that $100(i-\frac{1}{2})/n$ can be plotted on the ordinate corresponding to $\log\log[1-F(x)]^{-1}$ and the observed values can be plotted on the abscissa corresponding to $\log x$. Equation (3) can be written as

$$W = a+bz \qquad \ldots \quad (4)$$

where

$$W = \log\log[1-F(x)]^{-1}$$
$$z = \log x$$
$$b = \eta$$
$$a = -\eta\log\sigma$$

The estimates of the Weibull parameters from the probability plot are

$$\hat{\eta} = \hat{b} \qquad \ldots \quad (5)$$
$$\hat{\sigma} = \exp(-\hat{a}/\hat{b})$$

where $\hat{a}$ and b are the intercept and slope respectively of the best line fit.

c. Type I Extreme value distribution

The cumulative extreme value distribution function for the largest element is

$$F(x) = \exp\{-e^{-[(x-\mu)/\sigma]}\}, \; -\infty < x < \infty, \; -\infty < \mu < \infty, \; \sigma > 0$$

where μ and σ are location and scale parameters respectively. Thus the reduced variate

$$y = -\log\{-\log F(x)\} = \frac{x-\mu}{\sigma} \qquad \ldots \quad (6)$$

will plot as a straight line against observations from the distribution. Extreme value probability paper is so scaled that $[(i-\frac{1}{2})\,100/n]$, can be plotted directly against the values of the ordered observations. Equation (6) can be written as

$$y = a+bx$$

where $b = \frac{1}{\sigma}$ and $a = \frac{-\mu}{\sigma}$. The estimates of the parameters are

$$\hat{\sigma} = \frac{1}{\hat{b}} \text{ and } \hat{\mu} = -\frac{\hat{a}}{\hat{b}}$$

where $\hat{a}$ and b are the intercept and slope respectively to the best line fit.

a. One sample problem.

To test the hypothesis that a given sample $(x_1, x_2, \ldots, x_n)$ has arisen from a population with a numerically specified distribution function $F(x)$.

The Kolmogorov-Smirnov test (Table 10.1)

Let $F_n(x)$ be the proportion of observations in the sample less than or equal to x. $F_n(x)$ is called the empirical distribution function. Define

$$D^+(n) = \sup \{F_n(x) - F(x)\}$$
$$D^-(n) = \sup \{F(x) - F_n(x)\}$$
$$D(n) = \sup | F_n(x) - F(x) | = \max \{D^+(n), D^-(n)\}.$$

The choice of the test criterion depends on the specific departures intended to be detected.

The 1% and 5% critical values of $D^+(n)$, $D^-(n)$ and $D(n)$ are given in Table 10.1 for $n=1(1)\,20(5)35$ in the special case where $F(x)$ is continuous. A computed value of the criterion larger than or equal to the critical value given in Table 10.1 is significant. Table 10.1 also gives formulae for calculating the critical values when n is large.

Example. Test if the observations .068, .098, .117, .136, .317, .628 could have arisen in sampling from a rectangular distribution over the interval (0, 1).

Here $n = 6$, and $D(n) = .531$. The 5% value of $D(n)$ for $n = 6$ is 521. Hence the observed value is significant at the 5% level.

b. Two sample problem

Consider two samples $(x_{11}, x_{12}, \ldots, x_{1n_1})$ and $(x_{21}, x_{22}, \ldots, x_{2n_2})$ of size n_1 and n_2 respectively and the hypothesis that both the samples have arisen from the same population.

(i) *The Kolmogorov-Smirnov test* (Table 10.2)

Let F_{1n_1} and F_{2n_2} be the empirical distribution functions derived from samples 1 and 2 respectively. Define

$$D^+(n_1, n_2) = \sup \{F_{n_1}(x) - F_{n_2}(x)\}$$

$$D^-(n_1, n_2) = \sup \{F_{n_2}(x) - F_{n_1}(x)\}$$

$$D(n_1, n_2) = \sup \; |F_{n_1}(x) - F_{n_2}(x)|$$

$$= \max \{D^+(n_1, n_2), D^-(n_1, n_2)\}.$$

The choice of the test criterion depends on the specific departures from the hypothesis intended to be detected.

For the special case $n_1 = n_2 = n$, Table 10.2 provides 5% and 1% critical values for $n\ D^+(n, n)$ (or $nD^-(n, n)$) and $nD(n, n)$ covering the values of $n = 3(1)\ 30(5)\ 40$. A computed value of $nD^+(n, n)$ or $nD^-(n, n)$ or $nD(n, n)$ is declared to be significant if it exceeds or is equal to the critical value given in Table 10.2.

When n_1 and n_2 are large the following formulae may be used for calculating the critical values of the test criterion :

one-sided test statistic $D^+(n_1, n_2)$ or $D^-(n_1, n_2)$		two-sided test statistic $D(n_1, n_2)$	
1%	5%	1%	5%
$1.52\left(\frac{n_1+n_2}{n_1n_2}\right)^{\frac{1}{2}}$	$1.22\left(\frac{n_1+n_2}{n_1n_2}\right)^{\frac{1}{2}}$	$1.63\left(\frac{n_1+n_2}{n_1n_2}\right)^{\frac{1}{2}}$	$1.36\left(\frac{n_1+n_2}{n_1n_2}\right)^{\frac{1}{2}}$

The critical values given in Table 10.2 and also the asymptotic formulae given above are applicable only if the population distribution under the hypothesis is known to be continuous.

(ii) *Other tests*

Let the observations in the combined sample of size $n = n_1+n_2$ be serially arranged in increasing order of magnitude

$$x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}.$$

Let $i_1, i_2, \ldots, i_{n_2}$, $(1 \leqslant i_1 < i_2 < \ldots < i_{n_2} \leqslant n)$, be the serial orders of observations in sample 2.

A general form of test statistic for testing the hypothesis of equality of distribution functions is

$$a_n(i_1)+a_n(i_2)+ \ldots +a_n(i_{n_2})$$

where for each n, $a_n(i)$ is a given function defined over the integers $i = 1, 2, \ldots n$. The following are well known special cases :

(a) Fisher–Yates test

$a_n(i) =$ expected value of the i-th order statistic in a sample of size n from $N(0, 1)$. These expected values are given in Table 9.1.

(b) Wilcoxon (Mann–Whitney) test

$a_n(i) = i$.

(c) Van der Waerden test

$a_n(i) = \left(\frac{i}{n+1}\right)$-th quantile of $N(0, 1)$ defined by the equation

$$\int_{-\infty}^{a_n(i)} N(t)dt = \frac{i}{n+1}.$$

The values of $a_n(i)$ may be obtained by interpolation in Table 3.2.

(a) *The Fisher–Yates test* (Table 10.3).

Here observations in each sample are replaced by scores defined in the following manner. If a particular observation has rank i in the combined sample of size n, the score replacing this observation is given by the expected value of the i-th order statistic in a sample of size n from $N(0, 1)$. Define

C_1 = sum of the scores received by the second sample observations.

Table 10.3. provides the 1% and 5% critical values of C_1 for a two sided test and also the upper 1% and 5% values of C_1 for a one sided upper tail test. The lower 1% and 5% values are obtained by prefixing a negative sign to the upper 1% and 5% values respectively. Table 10.3 covers the values of $n = 6(1)10$ where n_1 is the size of the smaller sample.

For larger values of n one may apply the usual two sample t-test (described in 4) to the scores.

(b) *The Wilcoxon (Mann-Whitney) test* (Table 10.4)

Define U_{21} as the number of times an observation in the second sample precedes an observation in the first considering all pairs of observations one from each sample. Clearly

$$U_{21} = n_1 n_2 + \frac{n_2(n_2+1)}{2} - R_2$$

where R_2 = sum of the ranks assumed by the second sample observations. Define U_{12} in a similar manner and let $U = \min(U_{12}, U_{21})$.

Table 10.4 provides 1% and 5% critical values of U. An observed value equal to or less than the value given in table is declared to be significant. The selection of the statistic U_{12}, U_{21} or U depends upon the type of alternative hypotheses. For instance if it is desired to examine that the variable of the first population is stochastically larger than the second, one uses U_{12}. If the nature of departure to be detected is not specified one uses U Table 10.4 covers values of n_1 and $n_2 = 1(1)20$.

For larger values of n_1 and n_2, the sampling distribution of U may be assumed to be normal with

$$\text{mean} = n_1 n_2/2,$$
$$\text{variance} = n_1 n_2(n_1+n_2+1)/12.$$

(c) *The Wald-Wolfowitz run test* (Table 10.5)

Consider the serial arrangement of observations in increasing order of magnitude as discussed in (ii) above and replace each observation by 1 or 2 according as it arises from sample 1 or 2. A run is a succession of like symbols (numerals) preceded and followed by none or an unlike symbol (numeral). Let W be the total number of runs (i.e. the total of the number of runs of 1 and the number of runs of 2). W is proposed as a test statistic.

Table 10 5 provides the lower 1% and 5% critical values of W for n_1, n_2 upto 20.

For larger values of n_1, n_2 the sampling distribution of W may be assumed to be normal with

$$\text{mean} = \frac{2n_1n_2}{n_1+n_2}+1,$$

$$\text{variance} = \frac{2n_1n_2(2n_1n_2-n_1-n_2)}{(n_1+n_2)^2(n_1+n_2-1)}.$$

Example : In a certain feeding experiment 6 pigs were kept under a control diet while 6 others were provided with feed 'A' The gains in weight (in lbs) over a certain period were as follows :

Control : 8.1, 6.8, 6.9, 7.5, 8.8 (Sample 1)

A : 8.2, 8.4, 8.3, 8.7, 8.9 (Sample 2)

Examine if feed 'A' is an improvement over 'control'.

Kolmogorov-Smirnov test : Here $5D^+(5, 5) = 4$ which is equal to the 5% value given in Table 10.2. Hence the hypothesis that the two feeds are equally good is rejected (at the 5% level) in favour of 'A'.

Fisher-Yates test : We get the following rankings for the combined sample of 10 observations :

Rank order (i)	1	2	3	4	5	6	7	8	9	10
Value	6.8	6.9	7.5	8.1	8.2	8.3	8.4	8.7	8.8	8.9
Sample index	1	1	1	1	2	2	2	2	1	2
$Ex_{(i)}$ (from Table 9.1 for $n = 10$)					−.12	.12	.38	.66		1.54

Hence $C_1 = 2.58$. From Table 10.3 the 5% value of C_1 for a one-sided test (for $n = 10$ and $n_1 = 5$) is 2.58. The observed C_1 is thus significant at the 5% level.

Wilcoxon (Mann-Whitney) test : Here $R_2 = 36$. Hence $U_{21} = 25+15-36 = 4$. This is also significant at the 5% level, the critical value of U_{21} for $n_1 = n_2 = 5$ being 4 from Table 10.4.

Since $5D(5, 5)$ is also equal to 4 the hypothesis that the two feeds are equally good cannot be rejected by a two sided Kolmogorov– Smirnov test. It is seen that a two sided Fisher–Yates test or a two sided Wilcoxon (Mann-Whitney) test also fails to reject the hypothesis. When alternatives are two sided one could also use the Wald-Wolfowitz run test.

Wald-Wolfowitz run test : Total number of runs in the serial arrangement given above is 4. This is not significant at the 5% level, the critical value for $n_1 = n_2 = 5$ being 2 from Table 10.5.

c. Matched-pair sample

Consider n pairs of observations (x_i, y_i) $i = 1, 2, \ldots, n$ and the hypothesis that for each i the distribution of (x_i, y_i) is the same as that of (y_i, x_i)

(i) *The sign test*

Consider only the $n' \leqslant n$ pairs where $x_i \neq y_i$ and let r' be the number of pairs where $x_i < y_i$. For a given n' the distribution of r', under the given hypothesis, is binomial with $\pi = \frac{1}{2}$ This hypothesis could be tested in the manner discussed in 1.3.

(ii) *The Wilcoxon test* (Table 10.6)

Compute $d_i = x_i - y_i$. Here again as in the sign test all the $n - n'$ pairs where $d_i = 0$ are dropped out. The remaining d_i are ranked in increasing order of magnitude disregarding sign, the smallest $|d_i|$ receiving rank 1. Then to each rank is affixed the sign of d_i to which it corresponds. Define

T_- = sum of all ranks with a negative sign,

T_+ = sum of all ranks with a positive sign,

$T = \min \{T_-, T_+\}$.

Table 10.6 gives the 1% and 5% values of T, T_- or T_+. A computed value of T is significant if it is less than or equal to the value given in Table 10.6.

The choice of the statistic T_-, T_+ or T depends on the type of alternatives one wishes to detect.

Table 10.6 covers values of $n' = 6(1)25$. For larger values of n' the sampling distribution of T_+ (or equivalently T_-) may be assumed to be normal with

$$\text{mean} = \frac{n'(n'+1)}{4},$$

$$\text{variance} = \frac{n'(n'+1)(2n'+1)}{24}.$$

Note that
$$T_+ + T_- = \frac{n'(n'+1)}{2}.$$

Example: The following table gives the yield rate of paddy (in maunds per acre) as observed in ten pairs of concentric circles of radii 2 ft and 4 ft. Examine if the yield rate has been over-estimated by the smaller circle.

sample :	1	2	3	4	5	6	7	8	9	10	
2 ft.	6.12	5.39	5.59	6.34	6.29	5.98	5.61	4.43	5.93	5.33	(y)
4 ft.	5.50	6.00	4.71	6.12	5.93	5.56	5.41	5.14	5.66	5.67	(x)

Here we have

sample :	1	2	3	4	5	6	7	8	9	10
$x-y$	−0.62	0.61	−0.88	−0.22	−0.36	−0.42	−0.20	0.71	−0.27	0.34
Rank of $\|x-y\|$	8	7(+)	10	2	5	6	1	9(+)	3	4(+)

$T_+ = 4+9+7 = 20$. The 5% value of T_+ (for a one-sided test) for $n = 10$ is 10 from Table 10.6. Hence the observed result is not significant.

d. Spearman's rank correlation coefficient

When n individuals in a sample are ranked according to each of two different characteristics, the association between the characteristics may be measured by Spearman's rank correlation coefficient. This is the ordinary product moment correlation coefficient applied on rank pairs. When there are no tied ranks, the correlation coefficient can be computed by the formula

$$r_s = 1 - \frac{6 \sum_{1}^{n} d_i^2}{n^3 - n}$$

where d_i is the difference in the two ranks of the i-th individual.

Table 10.7 gives the upper 1% and 5% values of $|r_s|$ for a two-sided test and also the upper 1% and 5% values of r_s for a one-sided upper tail test. The lower 1% and 5% values of r_s for a one-sided lower tail test are obtained by prefixing a negative sign to the corresponding upper tail values.

Table 10.7 covers sample sizes upto $n = 10$. For n larger than 10, the critical values of r given in Table 7.1 with d.f. $\nu = n-2$, may be used as approximate critical values of r_s.

Example : A set of 10 individuals were ranked by two independent examiners with respect to their reasoning abilities. The ranks are given below. Test for association between ranks by the two examiners.

Individual :	1	2	3	4	5	6	7	8	9	10
Examiner 1 :	7	1	3	5	9	8	4	10	2	6
Examiner 2 :	6	2	4	3	8	10	5	9	1	7
d :	1	-1	-1	2	1	-2	-1	1	1	-1

$$\Sigma d_i^2 = 16,\ n^3 - n = 990,\ r_s = 1 - 96/990 = 0.9030$$

This is significant at the 1% level, the critical value for a two-sided test being 0.794 from Table 10.7.

TABLE 10.1. THE ONE SAMPLE KOLMOGOROV-SMIRNOV TEST

(5% and 1% critical values for one- and two-sided tests)

n	one-sided test $D^+(n)$ or $D^-(n)$ 1%	5%	two-sided $D(n)$ 1%	5%
1	.990	.950	.995	.975
2	.900	.776	.929	.842
3	.785	.636	.829	.708
4	.689	.565	.734	.624
5	.627	.509	.669	.563
6	.577	.468	.617	.519
7	.538	.436	.576	.483
8	.507	.410	.542	.454
9	.480	.387	.513	.430
10	.457	.369	.489	.409
11	.437	.352	.468	.391
12	.419	.338	.449	.375
13	.404	.325	.432	.361
14	.390	.314	.418	.349
15	.377	.304	.404	.338
16	.366	.295	.392	.327
17	.355	.286	.381	.318
18	.346	.279	.371	.309
19	.337	.271	.361	.301
20	.329	.265	.352	.294
25	.295	.238	.317	.264
30	.270	.218	.290	.242
35	.251	.202	.269	.224
over 35	$\frac{1.52}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$

TABLE 10.2. THE TWO SAMPLES KOLMOGOROV-SMIRNOV TEST

(5% and 1% critical values for one- and two-sided tests)

n	one sided $nD^+(n,n)$ or $nD^-(n,n)$ 1%	5%	two-sided $nD(n,n)$ 1%	5%
3	–	3	–	–
4	–	4	–	4
5	5	4	5	5
6	6	5	6	5
7	6	5	6	6
8	6	5	7	6
9	7	6	7	6
10	7	6	8	7
11	8	6	8	7
12	8	6	8	7
13	8	7	9	7
14	8	7	9	8
15	9	7	9	8
16	9	7	10	8
17	9	8	10	8
18	10	8	10	9
19	10	8	10	9
20	10	8	11	9
21	10	8	11	9
22	11	9	11	9
23	11	9	11	10
24	11	9	12	10
25	11	9	12	10
26	11	9	12	10
27	12	9	12	10
28	12	10	13	11
29	12	10	13	11
30	12	10	13	11
35	13	11	14	12
40	14	11	15	13
over 40	$1.52\sqrt{2n}$	$1.22\sqrt{2n}$	$1.63\sqrt{2n}$	$1.36\sqrt{2u}$

TABLE 10.3. THE FISHER-YATES TEST

(5% and 1% critical values for one- and two-sided tests)

$n = n_1+n_2$	n_1	one-sided 1%	5%	two-sided 1%	5%
6	3	—	2.11	—	—
7	2	—	2.11	—	—
7	3	—	2.46	—	—
8	2	—	2.27	—	—
8	3	—	2.42	—	2.74
8	4	—	2.59	—	2.89
9	2	—	2.42	—	—
9	3	—	2.33	—	2.69
9	4	3.26	2.42	—	2.72
10	2	—	2.54	—	2.54
10	3	3.20	2.32	—	2.66
10	4	3.32	2.54	3.58	2.82
10	5	3.46	2.58	3.70	2.94

TABLE 10.4. THE WILCOXON (MANN-WHITNEY) TEST

(1% critical values of U_{12} or U_{21} for one-sided test)

$n_1 \backslash n_2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
2	–	–	–	–	–	–	–	–	–	–	–	–	0	0	0	0	0	0	1	1	2
3	–	–	–	–	–	–	0	0	1	1	1	2	2	2	3	3	4	4	4	5	3
4	–	–	–	–	0	1	1	2	3	3	4	5	5	6	7	7	8	9	9	10	4
5	–	–	–	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	5
6	–	–	–	1	2	3	4	6	7	8	9	11	12	13	15	16	18	19	20	22	6
7	–	–	0	1	3	4	6	7	9	11	12	14	16	17	19	21	23	24	26	28	7
8	–	–	0	2	4	6	7	9	11	13	15	17	20	22	24	26	28	30	32	34	8
9	–	–	1	3	5	7	9	11	14	16	18	21	23	26	28	31	33	36	38	40	9
10	–	–	1	3	6	8	11	13	16	19	22	24	27	30	33	36	38	41	44	47	10
11	–	–	1	4	7	9	12	15	18	22	25	28	31	34	37	41	44	47	50	53	11
12	–	–	2	5	8	11	14	17	21	24	28	31	35	38	42	46	49	53	56	60	12
13	–	0	2	5	9	12	16	20	23	27	31	35	39	43	47	51	55	59	63	67	13
14	–	0	2	6	10	13	17	22	26	30	34	38	43	47	51	56	60	65	69	73	14
15	–	0	3	7	11	15	19	24	28	33	37	42	47	51	56	61	66	70	75	80	15
16	–	0	3	7	12	16	21	26	31	36	41	46	51	56	61	66	71	76	82	87	16
17	–	0	4	8	13	18	23	28	33	38	44	49	55	60	66	71	77	82	88	93	17
18	–	0	4	0	14	19	24	30	36	41	47	53	59	65	70	76	82	88	94	100	18
19	–	1	4	9	15	20	26	32	38	44	50	56	63	69	75	82	88	94	101	107	19
20	–	1	5	10	16	22	28	34	40	47	53	60	67	73	80	87	93	.00	107	114	20
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$n_2 \backslash n_1$

(5% critical values of U_{12} or U_{21} for one-sided test)

$n_1 \backslash n_2$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0	0	1
2	–	–	–	–	0	0	0	1	1	1	1	2	2	2	3	3	3	4	4	4	2
3	–	–	0	0	1	2	2	3	3	4	5	5	6	7	7	8	9	9	10	11	3
4	–	–	0	1	2	3	4	5	6	7	8	9	10	11	12	14	15	16	17	18	4
5	–	0	1	2	4	5	6	8	9	11	12	13	15	16	18	19	20	22	23	25	5
6	–	0	2	3	5	7	8	10	12	14	16	17	19	21	23	25	26	28	30	32	6
7	–	0	2	4	6	8	11	13	15	17	19	21	24	26	28	30	33	35	37	39	7
8	–	1	3	5	8	10	13	15	18	20	23	26	28	31	33	36	39	41	44	47	8
9	–	1	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	9
10	–	1	4	7	11	14	17	20	24	27	31	34	37	41	44	48	51	55	58	62	10
11	–	1	5	8	12	16	19	23	27	31	34	38	42	46	50	54	57	61	65	69	11
12	–	2	5	9	13	17	21	26	30	34	38	42	47	51	55	60	64	68	72	77	12
13	–	2	6	10	15	19	24	28	33	37	42	47	51	56	61	65	70	75	80	84	13
14	–	2	7	11	16	21	26	31	36	41	46	51	56	61	66	71	77	82	87	92	14
15	–	3	7	12	18	23	28	33	39	44	50	55	61	66	72	77	83	88	94	100	15
16	–	3	8	14	19	25	30	36	42	48	54	60	65	71	77	83	89	95	101	107	16
17	–	3	9	15	20	26	33	39	45	51	57	64	70	77	83	89	96	102	109	115	17
18	–	4	9	16	22	28	35	41	48	55	61	68	75	82	88	95	102	109	116	123	18
19	0	4	10	17	23	30	37	44	51	58	65	72	80	87	94	101	109	116	123	130	19
20	0	4	11	18	25	32	39	47	54	62	69	77	84	92	100	107	115	123	130	138	20
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	$n_2 \backslash n_1$

TABLE 10.4. (*continued*): THE WILCOXON (MANN-WHITNEY) TEST

(1% critical values of U for two-sided test)

n_1 \ n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
2	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	0	0	2
3	–	–	–	–	–	–	–	–	0	0	0	1	1	1	2	2	2	2	3	3	3
4	–	–	–	–	–	0	0	1	1	2	2	3	3	4	5	5	6	6	7	8	4
5	–	–	–	–	0	1	1	2	3	4	5	6	7	7	8	9	10	11	12	13	5
6	–	–	–	0	1	2	3	4	5	6	7	9	10	11	12	13	15	16	17	18	6
7	–	–	–	0	1	3	4	6	7	9	10	12	13	15	16	18	19	21	22	24	7
8	–	–	–	1	2	4	6	7	9	11	13	15	17	18	20	22	24	26	28	30	8
9	–	–	0	1	3	5	7	9	11	13	16	18	20	22	24	27	29	31	33	36	9
10	–	–	0	2	4	6	9	11	13	16	18	21	24	26	29	31	34	37	39	42	10
11	–	–	0	2	5	7	10	13	16	18	21	24	27	30	33	36	39	42	45	48	11
12	–	–	1	3	6	9	12	15	18	21	24	27	31	34	37	41	44	47	51	54	12
13	–	–	1	3	7	10	13	17	20	24	27	31	34	38	42	45	49	53	57	60	13
14	–	–	1	4	7	11	15	18	22	26	30	34	38	42	46	50	54	58	63	67	14
15	–	–	2	5	8	12	16	20	24	29	33	37	42	46	51	55	60	64	69	73	15
16	–	–	2	5	9	13	18	22	27	31	36	41	45	50	55	60	65	70	74	79	16
17	–	–	2	6	10	15	19	24	29	34	39	44	49	54	60	65	70	75	81	86	17
18	–	–	2	6	11	16	21	26	31	37	42	47	53	58	64	70	75	81	87	92	18
19	–	0	3	7	12	17	22	28	33	39	45	51	57	63	69	74	81	87	93	99	19
20	–	0	3	8	13	18	24	30	36	42	48	54	60	67	73	79	86	92	99	105	20
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	n_2 \ n_1

(5% critical values of U for a two-sided test)

n_1 \ n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	1
2	–	–	–	–	–	–	–	0	0	0	0	1	1	1	1	1	2	2	2	2	2
3	–	–	–	–	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	3
4	–	–	–	0	1	2	3	4	4	5	6	7	8	9	10	11	11	12	13	13	4
5	–	–	0	1	2	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20	5
6	–	–	1	2	3	5	6	8	10	11	13	14	16	17	19	21	22	24	25	27	6
7	–	–	1	3	5	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	7
8	–	0	2	4	6	8	10	13	15	17	19	22	24	26	29	31	34	36	38	41	8
9	–	0	2	4	7	10	12	15	17	20	23	26	28	31	34	37	39	42	45	48	9
10	–	0	3	5	8	11	14	17	20	23	26	29	33	36	39	42	45	48	52	55	10
11	–	0	3	6	9	13	16	19	23	26	30	33	37	40	44	47	51	55	58	62	11
12	–	1	4	7	11	14	18	22	26	29	33	37	41	45	49	53	57	61	65	69	12
13	–	1	4	8	12	16	20	24	28	33	37	41	45	50	54	59	63	67	72	76	13
14	–	1	5	9	13	17	22	26	31	33	40	45	50	55	59	64	67	74	78	83	14
15	–	1	5	10	14	19	24	29	34	39	44	49	54	59	64	70	75	80	85	90	15
16	–	1	6	11	15	21	26	31	37	42	47	53	59	64	70	75	81	86	92	98	16
17	–	2	6	11	17	22	28	34	39	45	51	57	63	67	75	81	87	93	99	105	17
18	–	2	7	12	18	24	30	36	42	48	55	61	67	74	80	86	93	99	106	112	18
19	–	2	7	13	19	25	32	38	45	52	58	65	72	78	85	92	99	106	113	119	19
20	–	2	8	13	20	27	34	41	48	55	62	69	76	83	90	98	105	112	119	127	20
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	n_2 \ n_1

TABLE 10.5. THE WALD-WOLFOWITZ RUN TEST

(1% critical values)

n_1 \ n_2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
3	–	–	–	–	–	–	–	–	–	2	2	2	2	2	2	2	2	2	3
4	–	–	–	–	–	2	2	2	2	2	2	2	3	3	3	3	3	3	4
5	–	–	–	2	2	2	2	3	3	3	3	3	3	3	3	4	4	4	5
6	–	–	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4	4	6
7	–	–	2	2	3	3	3	3	4	4	4	4	4	5	5	5	5	5	7
8	–	2	2	3	3	3	3	4	4	4	5	5	5	5	5	6	6	6	8
9	–	2	2	3	3	3	4	4	5	5	5	5	6	6	6	6	6	7	9
10	–	2	3	3	3	4	4	5	5	5	5	6	6	6	7	7	7	7	10
11	–	2	3	3	4	4	5	5	5	6	6	6	7	7	7	7	8	8	11
12	2	2	3	3	4	4	5	5	6	6	6	7	7	7	8	8	8	8	12
13	2	2	3	3	4	5	5	5	6	6	7	7	7	8	8	8	9	9	13
14	2	2	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	14
15	2	3	3	4	4	5	6	6	7	7	7	8	8	9	9	9	10	10	15
16	2	3	3	4	5	5	6	6	7	7	8	8	9	9	9	10	10	10	16
17	2	3	3	4	5	5	6	7	7	8	8	8	9	9	10	10	10	11	17
18	2	3	4	4	5	6	6	7	7	8	8	9	9	10	10	11	11	11	18
19	2	3	4	4	5	6	6	7	8	8	9	9	10	10	10	11	11	12	19
20	2	3	4	4	5	6	7	7	8	8	9	9	10	10	11	11	12	12	20
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	n_2 \ n_1

(5% critical values)

n_1 \ n_2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
2	–	–	–	–	–	–	–	–	–	–	2	2	2	2	2	2	2	2	2	2
3	–	–	–	–	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3
4	–	–	–	2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4	4
5	–	–	2	2	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5	5
6	–	2	2	3	3	3	3	4	4	4	4	5	5	5	5	5	5	6	6	6
7	–	2	2	3	3	3	4	4	5	5	5	5	5	6	6	6	6	6	6	7
8	–	2	3	3	3	4	4	5	5	5	6	6	6	6	6	7	7	7	7	8
9	–	2	3	3	4	4	5	5	5	6	6	6	7	7	7	7	8	8	8	9
10	–	2	3	3	4	5	5	5	6	6	7	7	7	7	8	8	8	8	9	10
11	–	2	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9	11
12	2	2	3	4	4	5	6	6	7	7	7	8	8	8	9	9	9	10	10	12
13	2	2	3	4	5	5	6	6	7	7	8	8	9	9	9	10	10	10	10	13
14	2	2	3	4	5	5	6	7	7	8	8	9	9	9	10	10	10	11	11	14
15	2	3	3	4	5	6	6	7	7	8	8	9	9	10	10	11	11	11	12	15
16	2	3	4	4	5	6	6	7	8	8	9	9	10	10	11	11	11	12	12	16
17	2	3	4	4	5	6	7	7	8	9	9	10	10	11	11	11	12	12	13	17
18	2	3	4	5	5	6	7	8	8	9	9	10	10	11	11	12	12	13	13	18
19	2	3	4	5	6	6	7	8	8	9	10	10	11	11	12	12	13	13	13	19
20	2	3	4	5	6	6	7	8	9	9	10	10	11	12	12	13	13	13	14	20
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	n_2 \ n_1

TABLE 10.6. THE WILCOXON MATCHED PAIR SIGNED RANK TEST

(5% and 1% critical values of T_-, T_+ and T)

n	T_- or T_+ (one-sided)		T (two-sided)	
	1%	5%	1%	5%
6	–	2	–	0
7	0	3	–	2
8	2	5	0	4
9	3	8	2	6
10	5	10	3	8
11	7	13	5	11
12	10	17	7	14
13	13	21	10	17
14	16	25	13	21
15	20	30	16	25
16	24	35	20	30
17	28	41	23	35
18	33	47	28	40
19	38	53	32	46
20	43	60	38	52
21	49	67	43	59
22	56	75	49	66
23	62	83	55	73
24	69	91	61	81
25	77	100	68	89

TABLE 10.7. SPEARMAN'S RANK CORRELATION COEFFICIENT

(5% and 1% critical values of r_s for one- and two-sided tests)

n	one-sided		two-sided	
	1%	5%	1%	5%
4	—	1.000	—	—
5	1.000	.900	—	1.000
6	.943	.829	1.000	.886
7	.893	.714	.929	.786
8	.833	.643	.881	.738
9	.783	.600	.833	.683
10	.746	.564	.794	.648

11.1. Measurements Data

a. Introduction

Control charts are used to detect changes in the mean value (centre of location) and in the variability (dispersion) of a *process*. The procedure consists in obtaining measurements on a sample of n items, computing chosen measures of location and dispersion, plotting the computed values on appropriate charts and taking decisions (regarding changes in the processes) depending on the positions of the plotted points.

How is a control chart drawn for any particular measure of location or dispersion ? Let T represent any such measure based on n observations in a sample. The *central line* of the control chart for T is drawn at $E(T)$, the expected value of T and *the upper and lower control limits* at $E(T)+b_1\sigma(T)$ and $E(T)-b_2\sigma(T)$ respectively, where b_1 and b_2 are suitably chosen constants and $\sigma(T)$ is the standard deviation of T. The limits obtained by choosing b_1, b_2 such that

$$\text{Probability } \{T-E(T) \geqslant b_1\sigma(T)\} = \alpha/2$$
$$\text{Probability } \{T-E(T) \leqslant -b_2\sigma(T)\} = \alpha/2$$

are called α *probability limits*. Those obtained by choosing $b_1 = b_2 = 3$ are called *three sigma* limits.

As an example for location, T may be the average $\bar{x}$ or the median $\tilde{x}$ of the sample. Charts using $\bar{x}$ and $\tilde{x}$ are called the $\bar{x}$ chart and $\tilde{x}$ (median) chart respectively. As a measure of dispersion T may be the standard deviation s, or the range R of the sample leading to an s-chart or an R-chart.

Let the true process average and standard deviation be represented by μ and σ. Under the asumption of normality of the observations, the $\sigma(T)$ for each measure T considered in Table 11.1 is found to be a multiple of σ, the process standard deviation. Hence the upper and lower control limits in all these cases can be written as

$$E(T)+z_1\sigma \quad E(T)-z_2\sigma$$

when μ and σ are specified.

The process mean and standard deviation may not be specified in practice, but may be estimable on the basis of previous data. If the past data are sufficiently numerous, yielding stable estimates of μ and σ, the same formulae $E(T)+z_1\sigma$, $E(T)-z_2\sigma$ for control limits can be used substituting estimates for $E(T)$ and σ.

b. Construction of a control chart

Table 11.1 provides the formulae for $E(T)$, and multipliers of σ or of an estimate of σ for a wide variety of measures, T. The general procedure for constructing a control chart is as follows :

(i) Decide on the *subgroup* (or sample) size n.

(ii) Choose a suitable measure of location and/or a measure of dispersion (see column (1) of Table 11.1 for measures commonly used).

(iii) (a) If the standards, i.e., the mean and the standard deviation of the process are known, use the formulae in column (3) for $E(T)$, the central line, and the formulae in column (6) for multiplying factors z_1, z_2. Thus, if we want a control chart for the measure s, the sample standard deviation, the central line is at $c_2\sigma$ and the upper and lower control limits are at $B_2\sigma$ and $B_1\sigma$.

(b) If the standards are not known, decide to use one of the alternative estimates of $E(T)$ given in columns (4) and (5) for the central line and one of the alternative estimates $\bar{s}$, $\bar{R}$, or $\tilde{R}$ for σ as defined in **11.1c** below. The multiplying factors for these estimates are given in columns (7), (8) and (9). Thus, if we want a control chart for the median $\tilde{x}$ choosing $\bar{\tilde{x}}$ as the estimate of μ and choosing $\bar{R}$ as an estimate of σ, the central line is at $\bar{\tilde{x}}$ and the formulae for the upper and lower control limits are, as found from column (8) of Table 11.1,

$$\bar{\tilde{x}}+F_2\bar{R} \text{ and } \bar{\tilde{x}}-F_2\bar{R}.$$

(iv) Having chosen the appropriate formula from Table 11.1 we have to find the numerical values of the symbols $A_1, A_2, \ldots, B_1, B_2, \ldots$ etc. They depend on the value of n and the nature of control limits required (3 sigma or probability limits). The values of all the symbols of Table 11.1 for 3 *sigma* limits are given in Table 11.2 for values of $n = 2$ (1) 10 and for some symbols upto $n = 20$. The values of some symbols for probability limits are given in Table 11.3.

c. Estimation of standards

The methods for computing different estimates of μ and σ from past data are as follows. Let $x_1 \ldots, x_N$ be the available series of past data. Divide the series into groups of n observations obtaining $k = [N/n]$ subgroups omitting if necessary a few observations at the end. It is assumed that N is large compared to n. For each subgroup compute the value of a measure of location and a measure of dispersion as shown in the following table.

In theory we can use any of the 8 estimates of μ in conjunction with any of the four estimates of σ, but in Table 11.1, we have indicated only some of the combinations for which tables exist for computing the control limits. It is also customary to examine the homogeneity of past data before using the estimated values of μ and σ for control limits. This is done by constructing control charts based on the estimates and plotting the subgroup values. Thus if we are computing the subgroup means and standard deviations we may construct an $\bar{x}$ chart using the estimates $\bar{\bar{x}}$ and $\bar{s}$. On such a chart we can plot the k consecutive values $\bar{x}_1, \ldots, \bar{x}_k$ and judge whether they were under control.

ESTIMATION OF STANDARDS FROM PAST DATA

sub-group no.	original series (past data)	alternative measures of location				alternative measures of dispersion	
		mean $\bar{x}$	median $\tilde{x}$	sum Σx	midrange M	standard deviation s	range R
1	x_1 ⋮ x_n	$\bar{x}_1$	$\tilde{x}_1$	$(\Sigma x)_1$	M_1	s_1	R_1
2	x_{n+1} ⋮ x_{2n}	$\bar{x}_2$	$\tilde{x}_2$	$(\Sigma x)_2$	M_2	s_2	R_2
⋮	⋮	⋮	⋮	⋮		⋮	⋮
k	$x_{n(k-1)+1}$ ⋮ x_{kn}	$\bar{x}_k$	$\tilde{x}_k$	$(\Sigma x)_k$	M_k	s_k	R_k
mean		$\bar{\bar{x}}$	$\bar{\tilde{x}}$	$\overline{(\Sigma x)}$	$\bar{M}$	$\bar{s}$	$\bar{R}$
median		$\tilde{\bar{x}}$	$\tilde{\tilde{x}}$	$(\tilde{\Sigma x})$	$\tilde{M}$	$\tilde{s}$	$\tilde{R}$
		providing 8 estimates of μ				providing 4 estimates of σ	

The symbols used are self-explanatory. Thus $\tilde{\tilde{x}}$ is the median of the subgroup medians $\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_k$; $\bar{\bar{x}}$ is the mean of subgroup means $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$; $\tilde{s}$ is the median of subgroup standard deviations $s_1, s_2, \ldots, s_k$ and so on.

11.2. Attributes Data

Instead of providing a measurement such as the length of an item, sometimes it is scored as bad or good, or as within or outside certain gauge limits, or as having a certain number of defects. The relevant formulae for the central line and the 3-sigma limits in such cases are given in Table 11.4.

d and p charts : When an item is scored as good or bad, the quality of a subgroup of n items is judged by the number defective (d) or the proportion defective (p). If the number defective is assumed to have a binomial distribution with the parameter π, then

$$E(p) = \pi, \quad E(d) = n\pi$$

$$\sigma(p) = \sqrt{\pi(1-\pi)/n}, \quad \sigma(d) = \sqrt{n\pi(1-\pi)}$$

which provide the formulae for the central line and the upper and lower control limits for the p and d charts.

If probability limits are required one has to use the cumulative probabilities of the binomial distribution. Let d_u and d_l denote the upper and lower limits for d at a probability $\alpha/2$ on each side. Then they satisfy the equations

$$\sum_{d \geqslant d_u} \binom{n}{d} \pi^d (1-\pi)^{n-d} \leqslant \frac{\alpha}{2}, \quad \sum_{d \leqslant d_l} \binom{n}{d} \pi^d (1-\pi)^{n-d} \leqslant \frac{\alpha}{2}.$$

The values of d_l and d_u for given n and π can be determined using the entries of Table 1.2. In the case of the p chart the upper and lower propability limits are d_u/n and d_l/n, where d_u and d_l are as determined above.

If the value of π is not specified, an estimate from past data may be substituted in the above formulae. The best estimate of π is p the observed proportion of defective items in the past data. Of course, the control chart for p or d with an estimated π can be used to test the homogeneity of past data by dividing the original series into subgroups of size n and plotting the individual values of p or d for each subgroup.

$b-a$ and $b+a$ or g and h charts In some cases, an item is scored as above an upper gauge value, as below a lower gauge value or as between the two values. Out of n items let b be the number of items above a given value and a be the number below another given value. The quality of subgroup is judged by $g = b-a$ which is sensitive for a change in the average size of the items and/or $h = b+a$ which is sensitive for a change in the dispersion of the size of the items. The formulae for the central line and upper and lower 3 sigma limits for g and h are given in Table 11.4, where π_1 and π_2 denote the hypothetical proportions of the items below the lower gauge and above the upper gauge value respectively

The determination of probability limits for small values of n is somewhat difficult in the case of $b-a$. For $b+a$ it is done as in the case of the number defective chart choosing $\pi = \pi_1+\pi_2$.

If the values of π_1 and π_2 are not known they may be estimated by p_1 and p_2, the observed proportions of items below the lower gauge value and above the upper gauge value respectively. The estimate of $\gamma(=\pi_2-\pi_1)$ is $\bar{g}(=p_2-p_1)$ and the estimate of $\delta(=\pi_1+\pi_2)$ is $\bar{p}(=p_1+p_2)$. The control charts constructed by using the estimated values of γ and δ can be used for testing the homogeneity of past data

11.3. Count of Defects Data

c, C, $\bar{c}$ charts : The quality of an item such as a glass pane or a piece of cloth of given dimensions is judged by the number of defects (c) on it. On the assumption of a Poisson distribution for c, the mean and variance are each equal to λ, the Poisson parameter. The formulae for the central line and the 3 sigma limits for c the number of defects on a single unit, C the total number of defects on n units and $\bar{c}$ the average number of defects per unit are given in Table 11.5. The probability limits can be obtained by first computing the cumulative probabilities from the individual terms of the Poisson distribution given in Table 2.1.

When the value of λ is not specified it may be estimated from past data by the average number of defects per unit. The homogeneity of past data can be examined by considering subgroups and plotting the successive values of C or $\bar{c}$ on the appropriate chart based on the estimated value of λ.

TABLE 11.1. FORMULAE FOR CONTROL CHART LINES: MEASUREMENTS DATA

Charts for central tendency and dispersion

(For description of estimates in columns (4), (5), (7), (8) and (9) see sub-section **11.1c**)

sub-group (sample) quality		central line			factors to multiply given standard or estimates to obtain UCL and LCL			
description of chart	symbol (statistic)	using given standard	using estimate		using given standard	using estimate		
	T		mean	median	σ	$\bar{s}$	$\bar{R}$	$\tilde{R}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
measures of location					*as distances from the central line*			
mean	$\bar{x}$	μ	$\bar{\bar{x}}$	$\tilde{\bar{x}}$	$\pm A$	$\pm A_1$	$\pm A_2$	$\pm A_3$
sum	Σx	$n\mu$	$(\overline{\Sigma x})$	$(\widetilde{\Sigma x})$	$\pm nA$	$\pm nA_1$	$\pm nA_2$	$\pm nA_3$
median	$\tilde{x}$	μ	$\bar{\tilde{x}}$	$\tilde{\tilde{x}}$	$\pm F$		$\pm F_2$	$\pm F_3$
midrange	M	μ	$\bar{M}$	$\tilde{M}$	$\pm G$		$\pm G_2$	$\pm G_3$
measures of dispersion					*as distances from the origin*			
standard deviation	s	$c_2\sigma$	$\bar{s}$	.	B_2 B_1	B_4 B_3	. .	
range	R	$d_2\sigma$	$\bar{R}$	$e_2\tilde{R}$	D_2 D_1	. .	D_4 D_3	D_6 D_5
moving range ($n = 2$)	r	1.128σ	$\bar{r}$	$1.183r$	$D_2(n = 2)$ $D_1(n = 2)$	.	$D_4(n = 2)$ $D_3(n = 2)$	$D_6(n = 2)$ $D_5(n = 2)$
order statistics					*as distances from the central line*			
largest measurement	L	$\mu+\frac{1}{2}d_2\sigma$	$\bar{M}+\frac{1}{2}\bar{R}$		$\pm H$	.	$\pm H_2$	
smallest measurement	S	$\mu-\frac{1}{2}d_2\sigma$	$\bar{M}-\frac{1}{2}\bar{R}$		$\pm H$	.	$\pm H_2$	
when L and S are plotted together with M		for UCL of L:			$+H'$	.	$+H'_2$	
		for LCL of S:			$-H'$	.	$-H'_2$	

Note n is the sub-group sample size. The 3 sigma values of all the symbols $A, A_1, \ldots, H_2, H'$ for different values of n are given in Table 11.2. The values of B_2, B_4, D_2, D_4, D_6 for one-sided upper probability limits at various levels are given in Table 11.3. The values of A, A_1, A_2, A_3, F, F_2, F_3, and G, G_2, G_3 for probability limits are obtained by multiplying the values for 3 sigma limits given in Table 11.2 by the following factors.

probability level:	0.1%	0.5%	1%	5%	10%
factor to multiply 3-sigma limits:	1.097	0.936	0.859	0.653	0.548

The values of the other symbols for probability limits are not given.

The values of c_2, d_2 and e_2 are also given in Table 11.2 for $n = 2(1)10$.

TABLE 11.2. FACTORS FOR COMPUTING CONTROL CHART LINES

Three sigma limits

factor	sub-group (sample) size n: 2	3	4	5	6	7	8	9	10	formula for general n
A	2.121	1.732	1.500	1.342	1.225	1.134	1.061	1.000	0.949	$3/\sqrt{n}$
A_1	3.760	2.394	1.880	1.596	1.410	1.277	1.175	1.094	1.028	A/c_2
A_2	1.881	1.023	0.729	0.577	0.483	0.419	0.373	0.337	0.308	A/d_2
A_3	2.224	1.091	0.758	0.594	0.495	0.429	0.380	0.343	0.314	A/d_m
B_1	0	0	0	0	0.026	0.105	0.167	0.219	0.262	c_2-3c
B_2	1.843	1.858	1.808	1.756	1.711	1.672	1.638	1.609	1.584	c_2+3c_3
B_3	0	0	0	0	0.030	0.118	0.185	0.239	0.284	B_1/c_2
B_4	3.267	2.568	2.266	2.089	1.970	1.882	1.815	1.761	1.716	B_2/c_2
D_1	0	0	0	0	0	0.204	0.388	0.547	0.687	d_2-3d_3
D_2	3.686	4.358	4.698	4.918	5.078	5.204	5.306	5.393	5.469	d_2+3d_3
D_3	0	0	0	0	0	0.076	0.136	0.184	0.223	D_1/d_2
D_4	3.267	2.575	2.282	2.115	2.004	1.924	1.864	1.816	1.777	D_2/d_2
D_5	0	0	0	0	0	0.078	0.139	0.187	0.227	D_1/d_m
D_6	3.864	2.744	2.375	2.179	2.055	1.967	1.902	1.850	1.808	D_2/d_m
F	2.121	2.009	1.638	1.607	1.390	1.376	1.230	1.223	1.116	$3\sigma_{\tilde{x}}$
F_2	1.880	1.187	0.796	0.691	0.549	0.509	0.432	0.412	0.363	F/d_2
F_3	2.224	1.265	0.828	0.712	0.562	0.520	0.441	0.419	0.369	F/d_m
G	2.121	1.805	1.638	1.532	1.458	1.402	1.358	1.322	1.292	$3\sigma_M$
G_2	1.880	1.067	0.796	0.659	0.575	0.518	0.477	0.445	0.420	G/d_2
G_3	2.224	1.137	0.828	0.679	0.590	0.530	0.487	0.453	0.427	G/d_m
H	2.477	2.244	2.104	2.007	1.935	1.878	1.832	1.793	1.760	$3\sigma_L$
H'	3.041	3.090	3.133	3.170	3.202	3.230	3.256	3.278	3.299	$H+\frac{1}{2}d_2$
H_2	2.195	1.326	1.022	0.863	0.763	0.694	0.643	0.604	0.572	H/d_2
H_2'	2.695	1.826	1.522	1.363	1.263	1.194	1.143	1.104	1.072	$H_2+\frac{1}{2}$
c_2	0.564	0.724	0.798	0.841	0.869	0.888	0.903	0.914	0.923	
d_2	1.128	1.693	2.059	2.326	2.534	2.704	2.847	2.970	3.078	
d_m	0.954	1.588	1.978	2.257	2.472	2.645	2.791	2.915	3.024	
e_2	1.183	1.066	1.041	1.031	1.025	1.022	1.020	1.019	1.018	

Note : The constants tabulated in Table 11.2 have been calculated under the assumption that the population distribution is normal. The constants in the general formula of the last column are defined as follows.

$$c_2 = E(s) = \sqrt{2}\,\Gamma\left(\frac{n}{2}\right) \div \sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right) \quad c_3 = \sigma_s = \left[\frac{n-1}{n} - c_2^2\right]^{\frac{1}{2}}, \; d_2 = E(R), \; d_3 = \sigma_R, \; d_m = E(\tilde{R}),$$

$e_2 = d_2/d_m$ where s, R, $\tilde{R}$, etc are as defined in column (2) of Table 11.1. In the tabulated values of d_m, $E(\tilde{R})$ is approximated by the median of the distribution of R.

TABLE 11.2 (*continued*). FACTORS FOR COMPUTING CONTROL CHART LINES

Three sigma limits

factor	sub-group (sample) size n									
	11	12	13	14	15	16	17	18	19	20
A	0.905	0.866	0.832	0.802	0.775	0.750	0.728	0.707	0.688	0.671
A_1	0.973	0.925	0.884	0.848	0.816	0.788	0.762	0.738	0.717	0.697
B_1	0.299	0.331	0.359	0.384	0.406	0.427	0.445	0.461	0.477	0.491
B_2	1.561	1.541	1.523	1.507	1.492	1.478	1.465	1.454	1.443	1.433
B_3	0.321	0.354	0.382	0.406	0.428	0.448	0.466	0.482	0.497	0.510
B_4	1.679	1.646	1.618	1.594	1.572	1.552	1.534	1.518	1.503	1.490

factor	21	22	23	24	25	26	27	28	29	30
A	0.655	0.640	0.626	0.612	0.600	0.588	0.577	0.567	0.557	0.548
A_1	0.679	0.662	0.647	0.632	0.619	0.606	0.594	0.583	0.572	0.562
B_1	0.504	0.516	0.527	0.538	0.548	0.557	0.566	0.574	0.582	0.589
B_2	1.424	1.415	1.407	1.399	1.392	1.385	1.378	1.372	1.366	1.360
B_3	0.523	0.534	0.545	0.555	0.565	0.574	0.582	0.590	0.597	0.604
B_4	1.477	1.466	1.455	1.445	1.435	1.426	1.418	1.410	1.403	1.396

TABLE 11.3. FACTORS FOR COMPUTING CONTROL CHART LINES

One-sided upper probability limits

probability level	factor	sub-group (sample) size n								
		2	3	4	5	6	7	8	9	10
0.1%	B_2	2.327	2.146	2.017	1.922	1.849	1.791	1.744	1.704	1.670
	B_4	4.125	2.966	2.528	2.286	2.129	2.016	1.932	1.865	1.810
	D_2	4.65	5.06	5.31	5.48	5.62	5.73	5.82	5.90	5.97
	D_4	4.12	2.99	2.58	2.36	2.22	2.12	2.04	1.99	1.94
	D_6	4.88	3.19	2.68	2.43	2.27	2.17	2.09	2.02	1.97
0.5%	B_2	1.985	1.879	1.792	1.724	1.671	1.628	1.592	1.562	1.536
	B_4	3.518	2.597	2.246	2.051	1.924	1.833	1.764	1.709	1.665
	D_2	3.97	4.42	4.69	4.89	5.03	5.15	5.26	5.34	5.42
	D_4	3.52	2.61	2.28	2.10	1.98	1.90	1.85	1.80	1.76
	D_6	4.16	2.78	2.37	2.17	2.04	1.95	1.88	1.83	1.79
1%	B_2	1.821	1.752	1.684	1.630	1.586	1.550	1.520	1.494	1.472
	B_4	3.228	2.421	2.111	1.939	1.826	1.745	1.684	1.635	1.595
	D_2	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16
	D_4	3.23	2.43	2.14	1.98	1.88	1.80	1.75	1.71	1.68
	D_6	3.82	2.59	2.22	2.04	1.93	1.84	1.79	1.74	1.71
5%	B_2	1.386	1.413	1.398	1.378	1.358	1.341	1.326	1.313	1.301
	B_4	2.457	1.953	1.752	1.639	1.563	1.510	1.469	1.437	1.410
	D_2	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47
	D_4	2.46	1.96	1.76	1.66	1.59	1.54	1.51	1.48	1.45
	D_6	2.90	2.08	1.83	1.71	1.63	1.58	1.54	1.51	1.48

Note: The values of B_4, D_4 and D_6 given in Table 11.3 provide only approximate probability limits. They have been calculated using the formulae $B_4 = B_2/c_2$, $D_4 = D_2/d_2$, $D_6 = D_2/d_m$.

TABLE 11.4. FORMULAE FOR CENTRAL LINE AND 3-SIGMA LIMITS : ATTRIBUTES DATA

sub-group (sample) quality		central line		upper and lower control limits UCL and LCL (as distances from central line)	
description of chart	symbol (statistic)	using given standard	using estimate	using given standard	using estimate
1. *Attributes Data—General* :—number defective *or* fraction defective chart :					
fraction defective	p	π	$\bar{p}$	$\pm 3\sqrt{\frac{\pi(1-\pi)}{n}}$	$\pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$
number defective	d	$n\pi$	$\bar{d}$ $(= n\bar{p})$	$\pm 3\sqrt{n\pi(1-\pi)}$	$\pm 3\sqrt{n\bar{p}(1-\bar{p})}$
2. *Attributes Data—double gauging* :—$(b-a)$ and $(b+a)$ charts : (The number below lower gauge is denoted by a and the number above upper gauge by b. The hypothetical proportion below the lower gauge is denoted by π_1 and above the upper gauge by π_2).					
change in location	$b-a = g$	$n(\pi_2-\pi_1) = n\gamma$	$\bar{g}$	$\pm 3\sqrt{n\delta - n\gamma^2}$	$\pm 3\sqrt{n\bar{p}-(\bar{g}^2/n)}$
change in dispersion	$b+a = h$	$n(\pi_1+\pi_2) = n\delta$	$\bar{h}$ $(= n\bar{p})$	$\pm 3\sqrt{n\delta(1-\delta)}$	$\pm 3\sqrt{n\bar{p}(1-\bar{p})}$

Note : n denotes the sub-group sample size.

TABLE 11.5. FORMULAE FOR CENTRAL LINE AND 3-SIGMA LIMITS : COUNT OF DEFECTS DATA

Number of defects or defects per unit charts

number of defects on unit $(n = 1)$	c	λ	$\bar{c}$	$\pm 3\sqrt{\lambda}$	$\pm 3\sqrt{\bar{c}}$
number of defects on group of n units	C $(= \Sigma c)$	$n\lambda$	$\bar{C}$ $(= n\bar{c})$	$\pm 3\sqrt{n\lambda}$	$\pm 3\sqrt{n\bar{c}}$
defects per unit	$\bar{c} = \frac{C}{n}$	λ	$\bar{\bar{c}}$	$\pm 3\sqrt{\frac{\lambda}{n}}$	$\pm 3\sqrt{\frac{\bar{\bar{c}}}{n}}$

Note : n denotes the sub-group sample size. The method for obtaining probability limits is explained in the text. They depend on the tables of individual terms of the binomial and Poisson distributions (see Tables 1.2 and 2.1).

11.4 Cumulative sum Control Charts

a. Introduction

The cumulative sum control chart (**cusum chart**) is used primarily to maintain current control of a process. Its advantage over the ordinary Shewhart chart is that it may be equally effective at less expense. This stems from the possibility of the cusum control chart picking up a sudden and a persistent change in the process average more rapidly than a comparable Shewhart chart, especially if the change is not large. The concept of Average Run Length (ARL) is used in the design of cusum charts. ARL is defined as the average number of samples plotted at a specified quality level before the chart indicates that the process is off target.

One sided decision interval scheme

Suppose we want to control the process at μ_0 and are interested in detecting changes in the process level in the upward direction. A reference value $k(> \mu_0)$ and a decision interval h are chosen, and the modified cusum is defined as follows. Compute successively

$$s_0 = 0, \quad s_r = \max\{0,\ s_{r-1}+(x_r-k)\} \qquad \dots \quad (1)$$

where x_r is the r-th observation. The chart indicates corrective action when for the first time $s_r \geqslant h$. If we want to detect shifts in the lower direction, we use

$$s_0 = 0,\ s_r = \min\{0,\ s_{r-1}+(x_r-k)\} \qquad \dots \quad (2)$$

where $k < \mu_0$. Corrective action is taken when for the first time $s_r \leqslant -h$.

Figure 1 is a nomogram which gives the ARL values for the control scheme (1) for any given value μ of the process level and chosen h, k when the characteristic is distributed normally with unit standard deviation. Suppose the process level is μ, process variability is σ and averages of samples of size n are plotted on the chart. We use L_a curve when $\mu < k$ and the L_r curve when $\mu > k$. We calculate $|k-\mu| \dfrac{\sqrt{n}}{\sigma}$ and $h \dfrac{\sqrt{n}}{\sigma}$ and locate these points on the line indicated by $|k-\mu| \dfrac{\sqrt{n}}{\sigma}$ and the curve indicated by $h \dfrac{\sqrt{n}}{\sigma}$ respectively and join them by a straight line. The point where this line cuts the L_a or the L_r curve as the case may be gives the ARL value.

Example

Given $\sigma = 10$, $n = 4$, $k = 105$ and $h = 13$, find the ARL values for $\mu = 100$, 102 and 110.

$$\frac{h\sqrt{n}}{\sigma} = 2.6,$$

$$|105-100| \frac{\sqrt{n}}{\sigma} = 1.0 \ \text{ARL} = 830 \text{ (from } L_a \text{ curve), for } \mu = 100.$$

$$|105-102| \frac{\sqrt{n}}{\sigma} = 0.6; \text{ ARL} = 104 \text{ (from } L_a \text{ curve)}, \quad \text{for } \mu = 102$$

$$|105-110| \frac{\sqrt{n}}{\sigma} = 1.0; \text{ ARL} = 3.5 \text{ (from } L_r \text{ curve)}, \quad \text{for } \mu = 110$$

Suppose we are given two values of the process level, say μ_0 acceptable level and μ_1 rejectable level, and also the desired ARL values i.e., of L_a and L_r respectively. L_a will be usually large and L_r will be small. There will be a number of combinations of n, k and h to meet these requirements. However, there are a number of advantages to be gained by the use of a central reference value i.e., $k = (\mu_0+\mu_1)/2$. We shall henceforth choose a reference value which is either central or near central. Table 11.6 which has been extracted from the nomogram gives the values of

$$|k-\mu_1| \frac{\sqrt{n}}{\sigma} = |k-\mu_0| \frac{\sqrt{n}}{\sigma} = |\mu_1-\mu_0| \frac{\sqrt{n}}{2\sigma}$$

and $h\sqrt{n}/\sigma$ for particular values of L_a and L_r when central reference value is used i.e., $k = (\mu_0+\mu_1)/2$. Then the values of n and h can be computed to design a suitable cusum control scheme.

Example: Design a suitable one sided decision interval scheme such that when $\mu_0 = 4.0$, $\mu_1 = 4.5$ and $\sigma = 1$, the values of L_a and L_r are 500 and 5 respectively

$$k = \frac{4.00+4.50}{2} = 4.25, \quad \frac{\mu_1-\mu_0}{2\sigma} = 0.25$$

From Table 11.6 we find that

$$\frac{(\mu_1-\mu_0)\sqrt{n}}{2\sigma} = 0.74 \quad \text{and} \quad \frac{h\sqrt{n}}{\sigma} = 3.18$$

From the first equation, $0.25\sqrt{n} = 0.74$ or $n = 8.76$ with the rounded value 9. Then

$$\frac{h\sqrt{n}}{\sigma} = 2.96h = 3.18 \text{ or } h = 1.07$$

V-mask Procedure :

V-mask procedure is used when one is interested in detecting the shifts from the target level μ_0 in either direction. The procedure is as follows. Compute

$$S_0 = 0, \; S_r = S_{r-1}+(x_r-\mu_0), \quad r = 1, 2, \ldots$$

and plot S_r against r. A *V*-mask is super imposed (see figure 2) on the chart with the vertex 0 at a distance d in horizontal plotting intervals ahead of the most recent point P on the chart. If the path of the chart cuts either limb of the *V*-mask, we conclude that the process is off the target. When the lower limb is cut, an increase in process level is indicated and when the upper limb is cut, a decrease in the process level is indicated. The parameters of the *V*-mask chart are

θ the half angle and d the lead distance. The parameters depend very much on the scale of the S_r axis and also on the horizontal distance between two successive plots. As such we define the scale factor w as the horizontal distance between successive points plotted on the chart measured in terms of unit distance on the vertical scale. It has been established that the V-mask procedure is equivalent to simultaneous application of two one-sided decision interval schemes. It is also shown that the ARL of the V-mask (L) is related to L_1 and L_2, the ARL's of two one sided decision interval schemes, by the formula

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$$

The relationship of the parameters of the V-mask and the two one sided decision interval schemes are given by

$$k_1 = \mu_0 + w \tan \theta, \quad h_1 = w\, d \tan \theta;$$
$$k_2 = \mu_0 - w \tan \theta, \quad -h_2 = -w\, d \tan \theta.$$

Hence either the nomogram or Table 11.6 can be used in the design of V-mask control schemes.

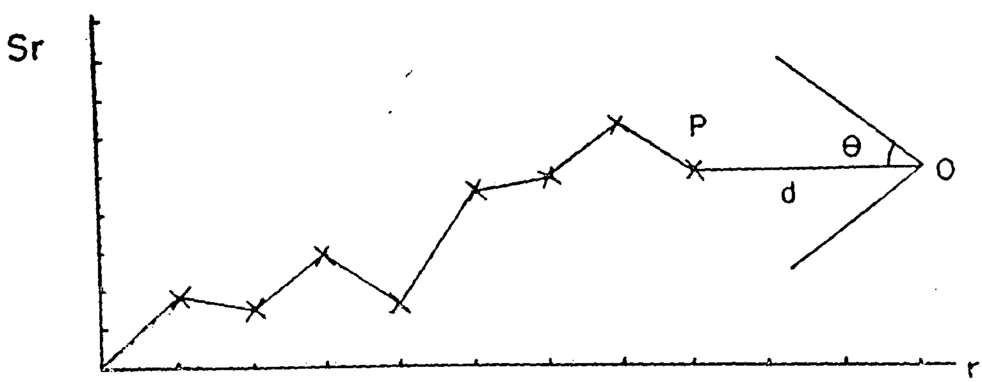


Fig. 2. V-mask Cusum Chart.

The procedure for finding the sample size, half angle θ and lead distance d is as follows :

The acceptable process level is μ_0 and we have two values of μ_1, the rejectable level : $\mu_1 = \mu_0 \pm \Delta$ where Δ is some positive constant. The values of ARL are specified to be L_a at μ_0 and L_r at $\mu_0 + \Delta$ and $\mu_0 - \Delta$.

(1) Read from Table 11.6 the values of $\left|\frac{\mu_1 - \mu_0}{2}\right| \frac{\sqrt{n}}{\sigma}$ and $\frac{h\sqrt{n}}{\sigma}$ for $2L_a$ and L_r.

(2) Since $\left|\frac{\mu_1 - \mu_0}{2}\right| = \Delta/2$ is known, the value of n is easily computed.

(3) $|k - \mu_0| = \left|\frac{\mu_1 - \mu_0}{2}\right| = w \tan \theta$ or $\theta = \tan^{-1}\left(\frac{|\mu_1 - \mu_0|}{2w}\right)$.

(4) $h = w\, d \tan \theta$ or $d = \frac{h\sqrt{n}}{\sigma} \div \frac{|\mu_1 - \mu_0|\sqrt{n}}{2\sigma}$.

Example : Devise a suitable V-mask control scheme, given that $\mu_0 = 4.0$, $\mu_1 = 4.0 \pm 0.5$, $\sigma = 1$ and $L_a = 500$ and $L_r = 5$.

Entering the Table 11.6 for $L_a = 1000 (= 2 \times 500)$ and $L_r = 5$, we get

$$\left| \frac{\mu_1 - \mu_0}{2\sigma} \right| \sqrt{n} = 0.80 \quad \text{and} \quad \frac{h\sqrt{n}}{\sigma} = 3.41.$$

Since $|\mu_1 - \mu_2|/2 = 0.25$, $\sigma = 1$, $0.25\sqrt{n} = 0.80$ or $n = 10.24$ with the rounded off value 11. When the scale factor $w = 1$, we get from Table 17.7.

$$\theta = \tan^{-1}\left(\frac{|\mu_1 - \mu_0|}{2w}\right) = \tan^{-1}\left(\frac{0.25}{w}\right) = \tan^{-1}(0.25) = 14°.$$

$$d = \frac{h\sqrt{n}}{\sigma} \div \frac{|\mu_1 - \mu_0|\sqrt{n}}{2\sigma} = \frac{3.41}{0.80} = 4.26 \text{ horizontal plotting intervals.}$$

Cusum charts for attributes

When we consider number of defects per sample or proportion defective (when the proportion is sufficiently small), we can use Poisson distribution for the design of cusum control schemes. Table 11.7 (Kemp, *Applied Statistics* **11**, 1962) is useful for this purpose. Let m_a and m_r be the acceptable and rejectable levels of the Poisson parameter. Table 11.7 gives for $R = m_r/m_a = 2.50$, 3.00, 3.50 and 4.00; $L_a = 500$, 250 and 125; $L_r = 5.0$, 7.5 and 10.0, the values of m_a, k and h. In case of proportion defective π we note that $m_a = n\pi_a$ and $m_r = n\pi_r$.

Example :

Given $\pi_a = 0.01$, $\pi_r = 0.04$, $L_a = 500$ and $L_r = 7.50$ design a suitable cusum scheme.

$R = 0.04/0.01 = 4$. For $R = 4$, $L_a = 500$, $L_r = 7.50$ we have $m_a = 0.24$, $k = 0.6$ and $h = 2.75$.

$$n \times \pi_a = m_a = 0.24 = n \times 0.01 \text{ or } n = 24.$$

Take samples of size 24 and let x_r be the number of defectives in the r-th sample. Define $s_0 = 0$, $s_r = \max\{0, s_{r-1} + (x_r - 0.6)\}$. Take corrective action when $s_r \geqslant 2.75$ for the first time.

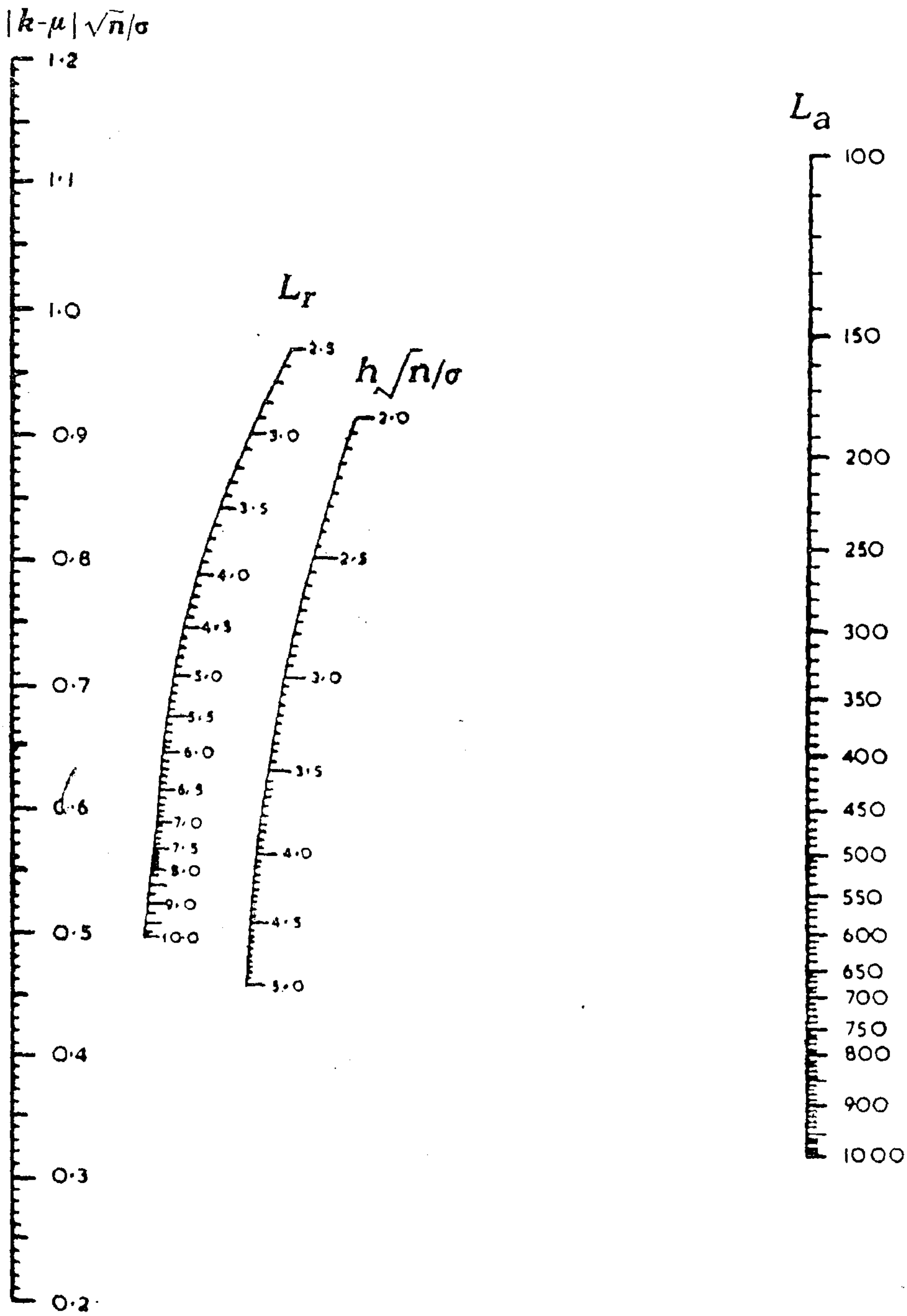


Fig. 1. Nomogram for determining the ARL when the quality characteristic is normally distributed.

TABLE 11.6. VALUES OF $\frac{|\mu_1-\mu_0|\sqrt{n}}{2\sigma}$ AND $\frac{h\sqrt{n}}{\sigma}$ FOR PARTICULAR VALUES OF L_a AND L_r.

L_r	L_a 200	500	1000
3	0.91	1.03	1.13
	2.07	2.27	2.38
4	0.76	0.85	0.92
	2.48	2.75	2.93
5	0.65	0.74	0.80
	2.86	3.18	3.41
6	0.58	0.66	0.72
	3.23	3.54	3.77
7	0.52	0.60	0.65
	3.45	3.80	4.08
8	0.48	0.55	0.60
	3.72	4.12	4.36
9	0.44	0.51	0.57
	3.89	4.30	4.67
10	0.42	0.48	0.53
	4.05	4.50	4.80

TABLE 11.7. VALUES OF m_a, R, h, AND k FOR FRACTION-DEFECTIVE SAMPLING SCHEMES

R	L_a	L_r 5.00 m_a	k	h	7.50 m_a	k	h	10.00 m_a	k	h
2.50	500	1.18	2.00	5.00	0.64	1.20	3.75	0.50	0.90	3.75
	250	0.93	1.50	4.50	0.52	0.90	3.50	0.42	0.80	3.00
	125	0.71	1.20	3.75	0.47	0.70	3.25	0.32	0.60	2.25
3.00	500	0.66	1.20	4.00	0.46	0.90	3.50	0.32	0.70	3.00
	250	0.56	0.90	3.00	0.40	0.80	3.00	0.27	0.60	2.50
	125	0.48	0.80	3.00	0.31	0.60	3.00	0.15	0.30	2.00
3.50	500	0.54	1.20	3.00	0.35	0.80	3.00	0.24	0.60	2.75
	250	0.41	0.90	2.50	0.27	0.60	2.50	0.18	0.40	2.50
	125	0.34	0.70	2.25	0.18	0.40	2.00	0.13	0.30	1.75
4.00	500	0.38	0.90	2.75	0.24	0.60	2.75	0.16	0.40	2.50
	250	0.32	0.80	2.25	0.21	0.60	2.00	0.12	0.30	2.00
	125	0.28	0.70	1.75	0.16	0.40	1.75	0.07	0.20	1.50

12.1 Confidence intervals for parameters.

a. Percentage defective

In sampling with replacement, the number d of defectives in a sample of size n from any lot follows a binomial distribution $b(n, \pi)$ where π is the proportion of defectives in the lot. This distribution also holds good, as an approximation, in sampling without replacement, if the lot size is very large compared to the sample size.

Confidence intervals for π are tabulated in Table 1.3 for $n \leqslant 30$. Table 12.1 provides $95^+\%$ and $99^+\%$ confidence intervals for 100π (percentage defective) based on the Clopper-Pearson system, for $n = 40, 50, 75, 100(100)500, 1000$.

b. Average number of defects

Under fairly general conditions, the number of defects per unit, in units of identical dimension, follows a Poisson distribution.

Two sided $95^+\%$ and $99^+\%$ confidence intervals for the Poisson mean λ, the average number of defects per unit, are given in Table 2.2.

c. Average measured value of a characteristic

When the measured value of a characteristic is normally distributed as $N(\mu, \sigma^2)$ confidence limits for its average (μ) based on a sample of size n are given by

$$95\% \text{ limits}: \quad \bar{x} \pm 1.96\sigma/\sqrt{n}$$
$$99\% \text{ limits}: \quad \bar{x} \pm 2.58\sigma/\sqrt{n}$$

if σ is known.

When σ is not known, the confidence limits for μ will be obtained from the following formula

$$100(1-\alpha)\% \text{ limits}: \quad \bar{x} \pm t_\alpha s/\sqrt{n-1}$$

where t_α is the two-sided $100\alpha\%$ point of the t-distribution with $n-1$ d.f. given in Table 4.1 (refer to the bottom row of Table 4.1), and $s = \sqrt{\Sigma(x_i-\bar{x})^2/n}$.

When σ is not known, instead of the sample standard deviation, the sample range R or the mean range $\bar{R}$ from k sub-groups (samples) each of size n may be used along with $\bar{x}$, to obtain confidence limits for μ. For computing 95% and 99% confidence intervals of the type $\bar{x} \pm h\ \bar{R}$, the factor h has been tabulated in Table 12.2 for $n = 2\ (1)15$ and $k = 1\ (1)15$.

d. Standard deviation of a measured value

Either the sample standard deviation s or the sample range R may be used to obtain the confidence interval for the parameter σ of the normal distribution. Table 12.3 gives factors f_1 and f_2 for computing 95% and 99% confidence intervals for σ, of the type $(f_1 s, f_2 s)$. Table 12.4 provides factors g_1 and g_2 for computing 95% and 99% confidence intervals for σ, of the type $(g_1 R, g_2 R)$.

Example. The range of breaking strength as observed in 10 pieces of hard drawn copper wire was 50.2 pounds. To obtain 95% confidence limits for σ.

From Table 12.4, the 95% values of g_1 and g_2 for $n = 10$ are read as 0.209 and 0.597 respectively. Hence 95% confidence limits for σ are $0.209 \times 50.2 = 10.5$ pounds and $0.597 \times 50.2 = 30.0$ pounds.

TABLE 12.1. CONFIDENCE INTERVALS FOR PERCENTAGE DEFECTIVE

Confidence coefficient: 95 percent

(n = sample size, d = observed number of defectives)

d	n: 40	50	75	100	200	300	400	500	1000	d
0	0.00— 8.81	0.00— 7.11	0.00— 4.80	0.00— 3.62	0.00— 1.83	0.00— 1.22	0.00— 0.92	0.00— 0.74	0.00— 0.37	0
1	0.06—13.16	0.05—10.65	0.03— 7.21	0.03— 5.45	0.01— 2.75	0.01— 1.84	0.01— 1.38	0.01— 1.11	0.00— 0.56	1
2	0.61—16.92	0.49—13.71	0.32— 9.30	0.24— 7.04	0.12— 3.57	0.08— 2.39	0.06— 1.79	0.05— 1.44	0.02— 0.72	2
3	1.57—20.39	1.25—16.55	0.83—11.25	0.62— 8.52	0.31— 4.32	0.21— 2.89	0.16— 2.18	0.12— 1.74	0.06— 0.87	3
4	2.79—23.66	2.22—19.23	1.47—13.10	1.10— 9.93	0.55— 5.04	0.36— 3.38	0.27— 2.54	0.22— 2.04	0.11— 1.02	4
5	4.19—26.80	3.33—21.81	2.20—14.88	1.64—11.28	0.80— 5.78	0.53— 3.88	0.40— 2.92	0.32— 2.34	0.16— 1.17	5
6	5.71—29.84	4.53—24.31	2.99—16.60	2.23—12.60	1.09— 6.46	0.73— 4.33	0.54— 3.26	0.43— 2.61	0.22— 1.31	6
7	7.34—32.78	5.82—26.74	3.84—18.29	2.86—13.89	1.40— 7.12	0.93— 4.77	0.70— 3.59	0.56— 2.88	0.28— 1.44	7
8	9.05—35.65	7.17—29.11	4.72—19.94	3.52—15.16	1.73— 7.76	1.15— 5.21	0.86— 3.92	0.69— 3.14	0.34— 1.58	8
9	10.84—38.45	8.58—31.44	5.64—21.56	4.20—16.40	2.07— 8.40	1.37— 5.64	1.03— 4.25	0.82— 3.40	0.41— 1.71	9
10	12.69—41.20	10.03—33.72	6.58—23.16	4.90—17.62	2.41— 9.03	1.60— 6.07	1.20— 4.57	0.96— 3.66	0.48— 1.84	10
11	14.60—43.89	11.53—35.96	7.56—24.73	5.62—18.83	2.77— 9.66	1.84— 6.49	1.37— 4.88	1.10— 3.92	0.55— 1.97	11
12	16.56—46.53	13.06—38.17	8.55—26.28	6.36—20.02	3.13—10.28	2.08— 6.90	1.55— 5.20	1.24— 4.17	0.62— 2.09	12
13	18.57—49.13	14.63—40.34	9.57—27.81	7.11—21.20	3.50—10.89	2.32— 7.32	1.74— 5.51	1.39— 4.42	0.69— 2.22	13
14	20.63—51.68	16.23—42.49	10.60—29.33	7.87—22.37	3.88—11.49	2.57— 7.73	1.92— 5.82	1.54— 4.67	0.77— 2.34	14
15	22.73—54.20	17.86—44.61	11.65—30.83	8.65—23.53	4.26—12.09	2.82— 8.13	2.11— 6.12	1.69— 4.91	0.84— 2.47	15
16	24.86—56.67	19.52—46.70	12.71—32.32	9.43—24.68	4.64—12.69	3.08— 8.53	2.30— 6.43	1.84— 5.16	0.92— 2.59	16
17	27.04—59.11	21.21—48.77	13.79—33.79	10.23—25.82	5.03—13.29	3.33— 8.94	2.49— 6.73	1.99— 5.40	0.99— 2.71	17
18	29.26—61.51	22.92—50.81	14.89—35.25	11.03—26.95	5.42—13.88	3.59— 9.33	2.69— 7.03	2.14— 5.64	1.07— 2.84	18
19	31.51—63.87	24.65—52.83	15.99—36.70	11.84—28.07	5.82—14.46	3.85— 9.73	2.88— 7.33	2.30— 5.88	1.15— 2.96	19
20	33.80—66.20	26.41—54.82	17.11—38.14	12.67—29.18	6.22—15.04	4.12—10.12	3.08— 7.63	2.46— 6.12	1.22— 3.08	20
21		28.19—56.79	18.24—39.56	13.49—30.29	6.62—15.62	4.38—10.52	3.28— 7.93	2.62— 6.36	1.30— 3.20	21
22		29.99—58.75	19.38—40.98	14.33—31.39	7.03—16.20	4.65—10.91	3.48— 8.22	2.78— 6.60	1.38— 3.32	22
23		31.81—60.68	20.53—42.38	15.17—32.49	7.44—16.78	4.92—11.30	3.68— 8.51	2.94— 6.83	1.46— 3.44	23
24		33.66—62.58	21.69—43.78	16.02—33.57	7.85—17.35	5.19—11.68	3.88— 8.81	3.10— 7.07	1.54— 3.55	24
25		35.53—64.47	22.86—45.17	16.88—34.66	8.26—17.92	5.47—12.07	4.08— 9.10	3.26— 7.30	1.62— 3.67	25
30			28.85—51.96	21.24—39.98	10.37—20.73	6.85—13.98	5.12—10.54	4.08— 8.46	2.03— 4.26	30
35			35.05—58.55	25.73—45.18	12.52—23.51	8.27—15.86	6.17—11.97	4.92— 9.61	2.45— 4.84	35
40				30.33—50.28	14.71—26.24	9.71—17.72	7.24—13.38	5.78—10.74	2.87— 5.41	40
45				35.03—55.27	16.93—28.94	11.16—19.56	8.33—14.77	6.64—11.86	3.30— 5.98	45
50				39.83—60.17	19.18—31.61	12.64—21.39	9.43—16.15	7.52—12.98	3.73— 6.54	50
60					23.77—36.88	15.63—24.99	11.65—18.89	9.29—15.18	4.61— 7.66	60
70					28.44—42.06	18.68—28.55	13.91—21.59	11.08—17.36	5.50— 8.76	70
80					33.19—47.16	21.76—32.06	16.20—24.27	12.90—19.52	6.40— 9.86	80
90					38.02—52.18	24.89—35.54	18.51—26.92	14.74—21.66	7.30—10.95	90
100					42.89—57.11	28.04—38.99	20.84—29.55	16.59—23.78	8.21—12.03	100
150						44.21—55.79	32.75—42.45	26.02—34.23	12.84—17.37	150
200							45.00—55.00	35.69—44.45	17.56—22.64	200
250								45.54—54.46	22.35—27.81	250
500									46.85—53.15	500

TABLE 12.1 (*continued*). CONFIDENCE INTERVALS FOR PERCENTAGE DEFECTIVE

Confidence coefficient: 99 percent

(n = sampl) size, d = observed number of defectives)

d	n: 40	50	75	100	200	300	400	500	1000	d
0	0.00—12.41	0.00—10.05	0.00— 6.82	0.00— 5.16	0.00— 2.61	0.00— 1.75	0.00— 1.32	0.00— 1.05	0.00— 0.53	0
1	0.01—17.15	0.01—13.94	0.01— 9.49	0.01— 7.20	0.00— 3.66	0.00— 2.45	0.00— 1.84	0.00— 1.48	0.00— 0.74	1
2	0.26—21.18	0.21—17.25	0.14—11.78	0.10— 8.94	0.05— 4.55	0.03— 3.05	0.03— 2.30	0.02— 1.84	0.01— 0.92	2
3	0.86—24.84	0.69—20.27	0.46—13.88	0.34—10.55	0.17— 5.38	0.11— 3.61	0.08— 2.72	0.07— 2.18	0.03— 1.09	3
4	1.73—28.26	1.38—23.11	0.91—15.85	0.68—12.06	0.34— 6.16	0.23— 4.14	0.17— 3.11	0.14— 2.50	0.07— 1.25	4
5	2.80—31.51	2.22—25.80	1.47—17.74	1.09—13.51	0.46— 6.84	0.30— 4.59	0.23— 3.46	0.18— 2.77	0.09— 1.39	5
6	4.02—34.63	3.19—28.40	2.10—19.57	1.56—14.92	0.69— 7.57	0.45— 5.08	0.34— 3.83	0.27— 3.07	0.14— 1.54	6
7	5.37—37.63	4.25—30.91	2.79—21.34	2.08—16.28	0.94— 8.28	0.62— 5.58	0.46— 4.19	0.37— 3.36	0.18— 1.69	7
8	6.82—40.54	5.39—33.35	3.53—23.06	2.63—17.61	1.21— 8.97	0.80— 6.03	0.60— 4.54	0.48— 3.64	0.24— 1.83	8
9	8.36—43.37	6.60—35.73	4.32—24.75	3.21—18.92	1.49— 9.65	0.99— 6.49	0.74— 4.89	0.59— 3.92	0.29— 1.97	9
10	9.98—46.12	7.86—38.05	5.14—26.40	3.82—20.20	1.79—10.32	1.19— 6.94	0.89— 5.23	0.71— 4.20	0.35— 2.11	10
11	11.68—48.81	9.19—40.32	5.99—28.03	4.45—21.45	2.10—10.98	1.39— 7.39	1.04— 5.57	0.83— 4.47	0.41— 2.25	11
12	13.44—51.43	10.56—42.55	6.88—29.63	5.10—22.70	2.42—11.64	1.60— 7.83	1.20— 5.90	0.96— 4.74	0.48— 2.38	12
13	15.26—54.00	11.97—44.74	7.78—31.20	5.77—23.92	2.75—12.28	1.82— 8.27	1.36— 6.23	1.08— 5.00	0.54— 2.51	13
14	17.13—56.51	13.42—46.89	8.71—32.75	6.45—25.13	3.08—12.92	2.04— 8.70	1.52— 6.56	1.22— 5.26	0.60— 2.65	14
15	19.06—58.97	14.91—49.00	9.67—34.29	7.15—26.32	3.42—13.55	2.26— 9.13	1.69— 6.88	1.35— 5.52	0.67— 2.78	15
16	21.05—61.38	16.44—51.08	10.64—35.80	7.87—27.51	3.77—14.18	2.49— 9.55	1.86— 7.20	1.49— 5.78	0.74— 2.91	16
17	23.08—63.74	18.00—53.12	11.63—37.30	8.59—28.68	4.12—14.80	2.72— 9.98	2.03— 7.52	1.62— 6.04	0.81— 3.04	17
18	25.16—66.05	19.59—55.14	12.64—38.78	9.33—29.84	4.48—15.42	2.96—10.39	2.21— 7.84	1.76— 6.29	0.88— 3.17	18
19	27.29—68.32	21.21—57.13	13.66—40.24	10.08—30.98	4.84—16.03	3.20—10.81	2.39— 8.15	1.91— 6.55	0.95— 3.30	19
20	29.46—70.54	22.87—59.08	14.70—41.69	10.84—32.12	5.21—16.63	3.44—11.22	2.57— 8.47	2.05— 6.80	1.02— 3.42	20
21		24.55—61.01	15.75—43.13	11.61—33.25	5.58—17.24	3.69—11.63	2.75— 8.78	2.20— 7.05	1.09— 3.55	21
22		26.26—62.91	16.82—44.55	12.39—34.37	5.96—17.84	3.93—12.04	2.94— 9.09	2.34— 7.30	1.16— 3.67	22
23		27.99—64.78	17.90—45.96	13.18—35.49	6.34—18.44	4.18—12.45	3.12— 9.39	2.49— 7.54	1.24— 3.80	23
24		29.76—66.63	19.00—47.36	13.97—36.59	6.72—19.03	4.43—12.85	3.31— 9.70	2.64— 7.79	1.31— 3.92	24
25		31.55—68.45	20.10—48.74	14.77—37.69	7.11—19.62	4.69—13.25	3.50—10.00	2.79— 8.03	1.39— 4.05	25
30			25.81—55.49	18.90—43.06	9.08—22.53	5.98—15.24	4.46—11.51	3.56— 9.25	1.77— 4.66	30
35			31.79—61.98	23.19—48.28	11.12—25.38	7.32—17.18	5.45—12.99	4.35—10.44	2.16— 5.26	35
40				27.63—53.35	13.20—28.18	8.68—19.10	6.47—14.44	5.15—11.61	2.56— 5.86	40
45				32.19—58.30	15.34—30.93	10.07—20.99	7.50—15.88	5.97—12.77	2.96— 6.45	45
50				36.89—63.11	17.51—33.65	11.48—22.86	8.55—17.30	6.81—13.92	3.37— 7.03	50
60					21.95—38.99	14.37—26.55	10.68—20.11	8.50—16.18	4.21— 8.19	60
70					26.51—44.21	17.32—30.16	12.86—22.87	10.23—18.42	5.06— 9.32	70
80					31.17—49.33	20.32—33.73	15.08—25.60	11.99—20.62	5.93—10.45	80
90					35.93—54.36	23.37—37.25	17.33—28.30	13.77—22.80	6.80—11.57	90
100					40.74—59.26	26.46—40.73	19.61—30.96	15.58—24.97	7.69—12.67	100
150						42.45—57.55	31.32—43.98	24.82—35.54	12.20—18.11	150
200							43.47—56.53	34.38—45.82	16.83—23.44	200
250								44.17—55.83	21.54—28.68	250
500									45.88—54.12	500

TABLE 12.2. FACTOR h FOR DETERMINING CONFIDENCE LIMITS FOR THE NORMAL MEAN USING RANGE OR MEAN RANGE

(Confidence coefficients : 95 per cent and 99 per cent)

n	P	k: 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0.95	6.36	1.72	1.08	0.83	0.70	0.61	0.55	0.50	0.46	0.44	0.41	0.39	0.37	0.36	0.34
	0.99	31.84	3.96	1.99	1.39	1.10	0.93	0.82	0.74	0.67	0.62	0.58	0.55	0.52	0.50	0.47
3	0.95	1.30	0.64	0.47	0.38	0.33	0.30	0.27	0.25	0.24	0.22	0.21	0.20	0.19	0.18	0.18
	0.99	3.01	1.05	0.71	0.56	0.48	0.42	0.38	0.35	0.33	0.31	0.29	0.27	0.26	0.25	0.24
4	0.95	0.72	0.41	0.31	0.26	0.23	0.21	0.19	0.18	0.17	0.16	0.15	0.14	0.14	0.13	0.13
	0.99	1.32	0.62	0.45	0.37	0.32	0.28	0.26	0.24	0.22	0.21	0.20	0.19	0.18	0.18	0.17
5	0.95	0.51	0.31	0.24	0.20	0.18	0.16	0.15	0.14	0.13	0.12	0.12	0.11	0.11	0.10	0.10
	0.99	0.84	0.45	0.34	0.28	0.24	0.22	0.20	0.19	0.17	0.16	0.16	0.15	0.14	0.14	0.13
6	0.95	0.40	0.25	0.20	0.17	0.15	0.13	0.12	0.11	0.11	0.10	0.10	0.09	0.09	0.09	0.08
	0.99	0.63	0.36	0.27	0.23	0.20	0.18	0.17	0.15	0.14	0.14	0.13	0.12	0.12	0.11	0.11
7	0.95	0.33	0.21	0.17	0.14	0.13	0.12	0.11	0.10	0.09	0.09	0.08	0.08	0.08	0.07	0.07
	0.99	0.51	0.30	0.23	0.19	0.17	0.16	0.14	0.13	0.12	0.12	0.11	0.11	0.10	0.10	0.09
8	0.95	0.29	0.19	0.15	0.13	0.11	0.10	0.09	0.09	0.08	0.08	0.07	0.07	0.07	0.07	0.06
	0.99	0.43	0.26	0.20	0.17	0.15	0.14	0.13	0.12	0.11	0.10	0.10	0.09	0.09	0.09	0.08
9	0.95	0.25	0.17	0.13	0.11	0.10	0.09	0.08	0.08	0.07	0.07	0.07	0.06	0.06	0.06	0.06
	0.99	0.37	0.23	0.18	0.15	0.14	0.12	0.11	0.11	0.10	0.09	0.09	0.09	0.08	0.08	0.08
10	0.95	0.23	0.15	0.12	0.10	0.09	0.08	0.08	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05
	0.99	0.33	0.21	0.16	0.14	0.12	0.11	0.10	0.10	0.09	0.09	0.08	0.08	0.07	0.07	0.07
11	0.95	0.21	0.14	0.11	0.10	0.08	0.08	0.07	0.07	0.06	0.06	0.06	0.05	0.05	0.05	0.05
	0.99	0.30	0.19	0.15	0.13	0 11	0.10	0.10	0.09	0.08	0.08	0.08	0.07	0.07	0.07	0.06
12	0.95	0.19	0.13	0.10	0.09	0.08	0.07	0.07	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.04
	0.99	0.28	0.18	0.14	0.12	0.11	0.10	0.09	0.08	0.08	0.07	0.07	0.07	0.06	0.06	0.06
13	0.95	0.18	0.12	0.10	0.08	0.07	0.07	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.04
	0.99	0.26	0.17	0.13	0.11	0.10	0.09	0.08	0.08	0.07	0.07	0.07	0.06	0.06	0.06	0.06
14	0.95	0.17	0.11	0.09	0.08	0.07	0.06	0.06	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04
	0.99	0.24	0.16	0.12	0.11	0.09	0.08	0.08	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05
15	0.95	0.16	0.11	0.09	0.07	0.07	0.06	0.06	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04
	0.99	0.22	0.15	0.12	0.10	0.09	0.08	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05	0.05

TABLE 12.3. FACTORS f_1 AND f_2 FOR DETERMINING CONFIDENCE LIMITS FOR NORMAL PARAMETER σ, USING SAMPLE STANDARD DEVIATION

sample size n	95 percent f_1	95 percent f_2	99 percent f_1	99 percent f_2
2	0.631	45.128	0.504	225.674
3	0.638	7.697	0.532	17.299
4	0.654	4.305	0.558	7.468
5	0.670	3.213	0.580	4.915
6	0.684	2.687	0.599	3.817
7	0.696	2.379	0.614	3.219
8	0.707	2.176	0.628	2.844
9	0.716	2.032	0.640	2.587
10	0.725	1.924	0.651	2.401
11	0.733	1.841	0.661	2.259
12	0.740	1.773	0.670	2.147
13	0.746	1.718	0.678	2.057
14	0.752	1.672	0.685	1.982
15	0.758	1.632	0.692	1.919
16	0.763	1.598	0.698	1.865
17	0.768	1.569	0.704	1.818
18	0.772	1.543	0.710	1.777
19	0.776	1.519	0.715	1.741
20	0.780	1.499	0.720	1.709
25	0.797	1.420	0.741	1.590
30	0.810	1.367	0.757	1.512
40	0.830	1.300	0.782	1.414
50	0.844	1.259	0.799	1.355
60	0.855	1.230	0.813	1.314
70	0.864	1.209	0.824	1.283
80	0.871	1.192	0.834	1.260
90	0.877	1.179	0.841	1.242
100	0.882	1.168	0.848	1.226

TABLE 12.4. FACTORS g_1 AND g_2 FOR DETERMINING CONFIDENCE LIMITS FOR NORMAL PARAMETER σ, USING SAMPLE RANGE

sample size n	95 percent g_1	95 percent g_2	99 percent g_1	99 percent g_2	sample size n
2	0.315	22.3	0.252	113.	2
3	0.272	3.30(1)	0.226	7.41(2)	3
4	0.251	1.68	0.213	2.92(3)	4
5	0.238	1.18	0.205	1.80	5
6	0.229	0.938	0.199	1.34	6
7	0.223	0.799	0.194	1.08	7
8	0.217	0.709	0.190	0.930	8
9	0.213	0.645	0.187	0.825	9
10	0.209	0.597	0.185	0.749	10
11	0.206	0.561	0.182	0.692	11
12	0.203	0.531	0.180	0.646	12
13	0.201	0.506	0.179	0.610	13
14	0.198	0.486	0.177	0.580	14
15	0.196	0.468	0.175	0.555	15
16	0.195	0.453	0.174	0.533	16
17	0.193	0.440	0.173	0.514	17
18	0.191	0.428	0.172	0.498	18
19	0.190	0.418	0.171	0.484	19
20	0.189	0.408	0.170	0.471	20

(1), (2), (3): These values could be in error in the last digit by the maximum amount of ± 1, ± 3, ± 1 respectively.

12.2 Tolerance Intervals

a. Introduction

Tolerance interval is constructed from experimental data such that the probability is p that at least a proportion P of the distribution will be enclosed by the interval. For the case of the normal population, Tables 12.5, 12.6, 12.7 and 12.8 give the appropriate factors for constructing tolerance intervals. Table 12.5 is to be used when s is taken as the estimate of σ. The desired limits are then $\bar{x} \pm ks$ where $\bar{x}$ is the sample mean. Table 12.5 gives the values of the factor k when $\bar{x}$ and s are computed from a sample of size N for $p = 0.75, 0.9, 0.95, 0.99$, $P = 0.75, 0.90, 0.95, 0.99, 0.999$; $N = 2(1)27$, $30(5)100(10)200(50)$ $300(100)1000$ and $N = \infty$. Table 12.6 is to be used when a single range is used for estimation of σ. Here we use $\bar{x}$ and R where $\bar{x}$ is the mean and R is the range in a sample of size N. The tolerance limits are constructed as $\bar{x} \pm k_1 R$. Table 12.6 gives the factor k_1 for the same values of p and P as in Table 12.5 and for $N = 2(1)20$. Table 12.7 is to be referred when we use the average range of samples of size 4. The tolerance interval is given by $\bar{\bar{x}} \pm k_2 \bar{R}$ where $\bar{\bar{x}}$ is the grand mean and $\bar{R}$ is the mean range in N samples of size 4. Table 12.7 gives the factor k_2 for $N = 4(1)20(5)30(10)50(25)$ 125 and $N = \infty$ for the same values of p and P as above. Table 12.8 is to be referred when mean range for samples of size 5 is used. The tolerance interval is given by $x \pm k_3 \bar{R}$ where $\bar{\bar{x}}$ is the grand mean and $\bar{R}$ is mean range in N samples of size 5. Table 12.8 gives the factor k_3 for $N = 4(1)20(5)30(10)50(25)100$ and $N = \infty$.

Table 12.5 is due to Bowker (*Techniques of Statistical Analysis*, Statistical Research Group, Columbia University, McGraw-Hill, New York, 1947). Tables 12.6, 12.7 and 12.8 are due to Mitra (*Journal of American Statistical Association*, **52**, 1957).

b. Application

The tolerance intervals are mainly used in quality control work for asserting with a given confidence that a certain minimum proportion of the manufactured products will have the quality characteristic value between these limits.

Example

A sample of 28 tins of hydrogenated oil were taken and net weight was measured (in gms) giving $\bar{x} = 1002$ and $s = 12$. Find tolerance limits having confidence coefficient 0.95 for 90% of the population.

For $n = 28$, $p = 0.95$, $P = 0.90$, we find from the Table 12.5, $k = 2.164$. Hence the tolerance limits are $1002 \pm 2.164 \times 12 = 976$ to 1028.

TABLE 12.5. TOLERANCE FACTORS FOR NORMAL DISTRIBUTION

Factor k such that the probability is p that at least a proportion P of the distribution will be included between $\bar{x} \pm ks$ where $\bar{x}$ and s are computed from a sample size N

	$p = 0.75$					$p = 0.90$					$p = 0.95$					$p = 0.99$				
N \ P	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
2	4.498	6.301	7.414	9.531	11.920	11.407	15.978	18.800	24.167	30.227	22.858	32.019	37.674	48.430	60.573	114.363	160.193	188.491	242.300	303.054
3	2.501	3.538	4.187	5.431	6.844	4.132	5.847	6.919	8.974	11.309	5.922	8.380	9.916	12.861	16.208	13.378	18.980	22.401	29.055	36.616
4	2.035	2.892	3.431	4.471	5.657	2.932	4.166	4.943	6.440	8.149	3.779	5.369	6.370	8.299	10.502	6.614	9.398	11.150	14.527	18.383
5	1.825	2.599	3.088	4.033	5.117	2.454	3.494	4.152	5.423	6.879	3.002	4.275	5.079	6.634	8.415	4.643	6.612	7.855	10.260	13.015
6	1.704	2.429	2.889	3.779	4.802	2.196	3.131	3.723	4.870	6.188	2.604	3.712	4.414	5.775	7.337	3.743	5.337	6.345	8.301	10.548
7	1.624	2.318	2.757	3.611	4.593	2.034	2.902	3.452	4.521	5.750	2.361	3.369	4.007	5.248	6.676	3.233	4.613	5.488	7.187	9.142
8	1.568	2.238	2.663	3.491	4.444	1.921	2.743	3.264	4.278	5.446	2.197	3.136	3.732	4.891	6.226	2.905	4.147	4.936	6.468	8.234
9	1.525	2.178	2.593	3.400	4.330	1.839	2.626	3.125	4.098	5.220	2.078	2.967	3.532	4.631	5.889	2.677	3.822	4.550	5.966	7.600
10	1.492	2.131	2.537	3.328	4.241	1.775	2.535	3.018	3.959	5.046	1.987	2.839	3.379	4.433	5.649	2.508	3.582	4.265	5.594	7.129
11	1.465	2.093	2.493	3.271	4.169	1.724	2.463	2.933	3.849	4.906	1.916	2.737	3.259	4.277	5.552	2.378	3.397	4.045	5.308	6.766
12	1.443	2.062	2.456	3.223	4.110	1.683	2.404	2.863	3.758	4.792	1.858	2.655	3.162	4.150	5.291	2.274	3.250	3.870	5.079	6.477
13	1.425	2.036	2.424	3.183	4.059	1.648	2.355	2.805	3.682	4.697	1.810	2.587	3.081	4.044	5.158	2.190	3.130	3.727	4.893	6.240
14	1.409	2.013	2.398	3.148	4.016	1.619	2.314	2.756	3.618	4.615	1.770	2.529	3.012	3.955	5.045	2.120	3.029	3.608	4.737	6.043
15	1.395	1.994	2.375	3.118	3.979	1.594	2.278	2.713	3.562	4.545	1.735	2.480	2.954	3.878	4.949	2.060	2.945	3.507	4.605	5.876
16	1.383	1.977	2.355	3.092	3.946	1.572	2.246	2.676	3.514	4.484	1.705	2.437	2.903	3.812	4.865	2.009	2.872	3.421	4.492	5.732
17	1.372	1.962	2.337	3.069	3.917	1.552	2.219	2.643	3.471	4.430	1.679	2.400	2.858	3.754	4.791	1.965	2.808	3.345	4.393	5.607
18	1.363	1.948	2.321	3.048	3.891	1.535	2.194	2.614	3.433	4.382	1.655	2.366	2.819	3.702	4.725	1.926	2.753	3.279	4.307	5.497
19	1.355	1.936	2.307	3.030	3.867	1.520	2.172	2.588	3.399	4.399	1.635	2.337	2.784	3.656	4.667	1.891	2.703	3.221	4.230	5.399
20	1.347	1.925	2.294	3.013	3.846	1.506	2.152	2.564	3.368	4.300	1.616	2.310	2.752	3.615	4.614	1.860	2.659	3.168	4.161	5.312
21	1.340	1.915	2.282	2.998	3.827	1.493	2.135	2.543	3.340	4.264	1.599	2.286	2.723	3.577	4.567	1.833	2.620	3.121	4.100	5.234
22	1.334	1.906	2.271	2.984	3.809	1.482	2.118	2.524	3.315	4.232	1.584	2.264	2.697	3.543	4.523	1.808	2.584	3.078	4.044	5.163
23	1.328	1.898	2.261	2.971	3.793	1.471	2.103	2.506	3.292	4.203	1.570	2.244	2.673	3.512	4.484	1.785	2.551	3.040	3.993	5.098
24	1.322	1.891	2.252	2.959	3.778	1.462	2.089	2.489	3.270	4.176	1.557	2.225	2.651	3.483	4.447	1.764	2.522	3.004	3.947	5.039
25	1.317	1.883	2.244	2.948	3.764	1.453	2.077	2.474	3.251	4.151	1.545	2.208	2.631	3.457	4.413	1.745	2.494	2.972	3.904	4.985
26	1.313	1.877	2.236	2.938	3.751	1.444	2.065	2.460	3.232	4.127	1.534	2.193	2.612	3.432	4.382	1.727	2.469	2.941	3.865	4.935
27	1.309	1.871	2.229	2.929	3.740	1.437	2.054	2.447	3.215	4.106	1.523	2.178	2.595	3.409	4.353	1.711	2.446	2.914	3.828	4.888
30	1.297	1.855	2.210	2.904	3.708	1.417	2.025	2.413	3.170	4.049	1.497	2.140	2.549	3.350	4.278	1.668	2.385	2.841	3.733	4.768
35	1.283	1.834	2.185	2.871	3.667	1.390	1.988	2.368	3.112	3.974	1.462	2.090	2.490	3.272	4.179	1.613	2.306	2.748	3.611	4.611
40	1.271	1.818	2.166	2.846	3.635	1.370	1.959	2.334	3.066	3.917	1.435	2.052	2.445	3.213	4.104	1.571	2.247	2.677	3.518	4.493

TABLE 12.5. (*contd.*) TOLERANCE FACTORS FOR NORMAL DISTRIBUTION

	$p = 0.75$					$p = 0.90$					$p = 0.95$					$p = 0.99$				
$N \backslash P$	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
45	1.262	1.805	2.150	2.826	3.609	1.364	1.935	2.306	3.030	3.871	1.414	2.021	2.408	3.165	4.042	1.539	2.200	2.621	3.444	4.399
50	1.255	1.794	2.138	2.809	3.588	1.340	1.916	2.284	3.001	3.833	1.396	1.996	2.379	3.126	3.993	1.512	2.162	2.576	3.385	4.323
55	1.249	1.785	2.127	2.795	3.571	1.329	1.901	2.265	2.976	3.801	1.382	1.976	2.354	3.094	3.951	1.490	2.130	2.538	3.335	4.260
60	1.243	1.778	2.118	2.784	3.556	1.320	1.887	2.248	2.955	3.774	1.369	1.958	2.333	3.066	3.916	1.471	2.103	2.506	3.293	4.206
65	1.239	1.771	2.110	2.773	3.543	1.312	1.875	2.235	2.937	3.751	1.359	1.943	2.315	3.042	3.886	1.455	2.080	2.478	3.257	4.160
70	1.235	1.765	2.104	2.764	3.531	1.304	1.865	2.222	2.920	3.730	1.349	1.929	2.299	3.021	3.859	1.440	2.060	2.454	3.225	4.120
75	1.231	1.760	2.098	2.757	3.521	1.298	1.856	2.211	2.906	3.712	1.341	1.917	2.285	3.002	3.835	1.428	2.042	2.433	3.197	4.084
80	1.228	1.756	2.092	2.749	3.512	1.292	1.848	2.202	2.894	3.696	1.334	1.907	2.272	2.986	3.814	1.417	2.026	2.414	3.173	4.053
85	1.225	1.752	2.087	2.743	3.504	1.287	1.841	2.193	2.882	3.682	1.327	1.897	2.261	2.971	3.795	1.407	2.012	2.397	3.150	4.024
90	1.223	1.748	2.083	2.737	3.497	1.283	1.834	2.185	2.872	3.669	1.321	1.889	2.251	2.958	3.778	1.398	1.999	2.382	3.130	3.999
95	1.220	1.745	2.079	2.732	3.490	1.278	1.828	2.178	2.863	3.657	1.315	1.881	2.241	2.945	3.763	1.390	1.987	2.368	3.112	3.976
100	1.218	1.742	2.075	2.727	3.484	1.275	1.822	2.172	2.854	3.646	1.311	1.874	2.233	2.934	3.748	1.383	1.977	2.355	3.096	3.954
110	1.214	1.736	2.069	2.719	3.473	1.268	1.813	2.160	2.839	3.626	1.302	1.861	2.218	2.915	3.723	1.369	1.958	2.333	3.066	3.917
120	1.211	1.732	2.063	2.712	3.464	1.262	1.804	2.150	2.826	3.610	1.294	1.850	2.205	2.898	3.702	1.358	1.942	2.314	3.041	3.885
130	1.208	1.728	2.059	2.705	3.456	1.257	1.797	2.141	2.814	3.595	1.288	1.841	2.194	2.883	3.683	1.349	1.928	2.298	3.019	3.857
140	1.206	1.724	2.054	2.700	3.449	1.252	1.791	2.134	2.804	3.582	1.282	1.833	2.184	2.870	3.666	1.340	1.916	2.283	3.000	3.833
150	1.204	1.721	2.051	2.695	3.443	1.248	1.785	2.127	2.795	3.571	1.277	1.825	2.175	2.859	3.652	1.332	1.905	2.270	2.983	3.811
160	1.202	1.718	2.047	2.691	3.437	1.245	1.780	2.121	2.787	3.561	1.272	1.819	2.167	2.848	3.638	1.326	1.896	2.259	2.968	3.792
170	1.200	1.716	2.044	2.687	3.432	1.242	1.775	2.116	2.780	3.552	1.268	1.813	2.160	2.839	3.627	1.320	1.887	2.248	2.955	3.774
180	1.198	1.713	2.042	2.683	3.427	1.239	1.771	2.111	2.774	3.543	1.264	1.808	2.154	2.831	3.616	1.314	1.879	2.239	2.942	3.759
190	1.197	1.711	2.039	2.680	3.423	1.236	1.767	2.106	2.768	3.536	1.261	1.803	2.148	2.823	3.606	1.309	1.872	2.230	2.931	3.744
200	1.195	1.709	2.037	3.677	3.419	1.234	1.764	2.102	2.762	3.529	1.258	1.798	2.143	2.816	3.597	1.304	1.865	2.222	2.921	3.731
250	1.190	1.702	2.028	2.665	3.404	1.224	1.750	2.085	2.740	3.501	1.245	1.780	2.121	2.788	3.561	1.286	1.839	2.191	2.880	3.678
300	1.186	1.696	2.021	2.656	3.393	1.217	1.740	2.073	2.725	3.481	1.236	1.767	2.106	2.767	3.535	1.273	1.820	2.169	2.850	3.641
400	1.181	1.688	2.012	2.644	3.378	1.207	1.726	2.057	2.703	3.453	1.223	1.749	2.084	2.739	3.499	1.255	1.794	2.138	2.809	3.589
500	1.177	1.683	2.006	2.636	3.368	1.201	1.717	2.046	2.689	3.434	1.215	1.737	2.070	2.721	3.475	1.243	1.777	2.117	2.783	3.555
600	1.175	1.680	2.002	2.631	3.360	1.196	1.710	2.038	2.678	3.421	1.209	1.729	2.060	2.707	2.458	1.234	1.764	2.102	2.763	3.530
700	1.173	1.677	1.998	2.626	3.355	1.192	1.705	2.032	2.670	3.411	1.204	1.722	2.052	2.697	3.445	1.227	1.755	2.091	2.748	3.511
800	1.171	1.675	1.996	2.623	3.350	1.189	1.701	2.027	2.663	3.402	1.201	1.717	2.046	2.688	3.434	1.222	1.747	2.082	2.736	3.495
900	1.170	1.673	1.993	2.620	3.347	1.187	1.697	2.023	2.658	3.396	1.198	1.712	2.040	2.682	3.426	1.218	1.741	2.075	2.726	3.483
1000	1.169	1.671	1.992	2.617	3.344	1.185	1.695	2.019	2.654	3.390	1.195	1.709	2.036	2.676	3.418	1.214	1.736	2.068	2.718	3.472
∞	1.150	1.645	1.960	2.576	3.291	1.150	1.645	1.960	2.576	3.291	1.150	1.645	1.960	2.576	3.291	1.150	1.645	1.960	2.576	3.291

TABLE 12.6. TOLERANCE FACTORS FOR NORMAL DISTRIBUTION

Factor k_1 such that the probability is p that at least a proportion P of the distribution will be included between $\bar{x} \pm k_1 R$ where $\bar{x}$ is the mean and R is the range in a sample of size N

	$p = 0.75$					$p = 0.90$					$p = 0.95$					$p = 0.99$				
N \ P	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
2	3.181	4.456	5.243	6.740	8.429	8.065	11.298	13.294	17.090	21.374	16.158	22.635	26.634	34.238	42.821	80.972	113.429	133.469	171.576	214.588
3	1.312	1.857	2.197	2.850	3.591	2.169	3.069	3.631	4.711	5.936	3.109	4.399	5.206	6.752	8.509	7.034	9.951	11.776	16.275	19.249
4	.916	1.301	1.544	2.012	2.546	1.321	1.877	2.227	2.902	3.672	1.704	2.422	2.873	3.744	4.737	2.978	4.233	5.021	6.543	8.279
5	.744	1.060	1.259	1.644	2.086	1.003	1.428	1.697	2.216	2.812	1.228	1.749	2.078	2.715	3.444	1.903	2.709	3.219	4.205	5.335
6	.647	.923	1.097	1.435	1.824	.837	1.194	1.420	1.857	2.360	.995	1.418	1.686	2.206	2.803	1.433	2.042	2.429	3.178	4.038
7	.584	.834	.992	1.299	1.652	.735	1.050	1.248	1.635	2.080	.856	1.222	1.453	1.903	2.420	1.176	1.678	1.996	2.615	3.325
8	.540	.771	.917	1.202	1.530	.666	.951	1.131	1.483	1.888	.764	1.090	1.297	1.700	2.165	1.015	1.449	1.724	2.261	2.878
9	.507	.723	.861	1.129	1.438	.615	.879	1.046	1.372	1.747	.698	.997	1.187	1.556	1.981	.903	1.290	1.536	2.014	2.565
10	.481	.687	.817	1.072	1.366	.577	.824	981	1.286	1.639	.648	.926	1.103	1.446	1.843	.823	1.176	1.400	1.836	2.340
11	.460	.657	.782	1.026	1.308	.546	.780	.929	1.219	1.554	.610	.871	1.037	1.361	1.735	762	1.088	1.296	1.701	2.168
12	.442	.632	.753	.988	1.260	.521	745	.887	1.164	1.484	.578	.827	.985	1.292	1.648	714	1.020	1.215	1.594	2.033
13	.428	.611	.728	.956	1.219	.501	.715	.852	1.118	1.426	.553	.790	.940	1.235	1.575	.675	964	1.148	1.507	1.922
14	.415	.594	.707	.928	1.184	.483	.690	.822	1.079	1.377	.531	.759	.904	1.187	1.514	.642	917	1.093	1.435	1.830
15	.405	.578	.689	.904	1.154	.468	.669	.797	1.046	1.334	.513	.733	.873	1.146	1.462	.614	.878	1.040	1.373	1.753
16	.395	.565	.673	.883	1.127	.455	.650	.774	1.016	1.297	.497	.710	.845	1.110	1.417	.591	.845	1.007	1.322	1.687
17	.386	.553	.658	.864	1.103	.443	.633	.755	.991	1.265	.482	.690	.822	1.109	1.377	.571	.816	.972	1.277	1.630
18	.379	.542	.645	.848	1.082	.433	.619	.737	.968	1.235	.470	.672	.801	1.051	1.342	.553	.790	941	1.236	1.578
19	.372	.532	.634	.833	1.063	.424	.605	.721	.947	1.209	.459	.656	.782	1.027	1.311	.538	.768	916	1.203	1.535
20	.366	.523	.623	.819	1.045	.415	.594	.707	.929	1.186	.449	.642	.765	1.065	1.282	.524	.748	892	1.171	1.495

TABLE 12.7. TOLERANCE FACTORS FOR NORMAL DISTRIBUTION

Factor k_2 such that the probability is p that at least a proportion P of the distribution will be included between $\bar{\bar{x}} \pm k_2\bar{R}$ where $\bar{\bar{x}}$ is the grand mean and $\bar{R}$ is the mean range in N samples of size 4

	$p = 0.75$					$p = 0.90$					$p = 0.95$					$p = 0.99$				
N \ P	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
4	.670	.958	1.141	1.498	1.912	.767	1.097	1.307	1.716	2.190	.833	1.191	1.418	1.862	2.377	.971	1.388	1.653	2.171	2.770
5	.656	.937	1.116	1.466	1.872	.740	1.058	1.261	1.656	2.114	.797	1.139	1.357	1.783	2.275	.915	1.308	1.558	2.047	2.613
6	.645	.922	1.099	1.444	1.843	.721	1.031	1.228	1.613	2.060	.771	1.103	1.313	1.726	2.203	.876	1.252	1.491	1.959	2.502
7	.637	.911	1.086	1.426	1.821	.706	1.010	1.203	1.581	2.019	.752	1.075	1.281	1.683	2.149	.846	1.210	1.441	1.894	2.418
8	.631	.903	1.075	1.413	1.806	.695	.995	1.184	1.556	1.988	.737	1.054	1.255	1.650	2.108	.823	1.177	1.402	1.842	2.354
9	.626	.895	1.067	1.402	1.791	.686	.980	1.168	1.535	1.961	.725	1.036	1.235	1.622	2.073	.804	1.150	1.370	1.801	2.300
10	.622	.890	1.060	1.393	1.780	.678	.969	1.155	1.518	1.939	.714	1.021	1.217	1.600	2.043	.789	1.128	1.344	1.766	2.256
11	.619	.885	1.054	1.385	1.770	.671	.960	1.144	1.503	1.921	.706	1.009	1.202	1.580	2.019	.776	1.109	1.321	1.737	2.218
12	.616	.880	1.049	1.379	1.761	.666	.952	1.134	1.491	1.905	.698	.999	1.190	1.564	1.998	.764	1.093	1.302	1.712	2.187
13	.613	.877	1.044	1.373	1.753	.661	.945	1.126	1.480	1.890	.692	.989	1.179	1.549	1.979	.755	1.079	1.286	1.689	2.158
14	.611	.873	1.041	1.368	1.747	.657	.939	1.119	1.470	1.878	.686	.981	1.169	1.537	1.963	.746	1.067	1.271	1.671	2.134
15	.609	.870	1.037	1.363	1.741	.653	.933	1.112	1.462	1.867	.681	.974	1.161	1.525	1.948	.738	1.056	1.258	1.654	2.112
16	.607	.868	1.034	1.359	1.736	.649	.928	1.106	1.454	1.857	.677	.967	1.153	1.515	1.935	.732	1.046	1.246	1.638	2.093
17	.605	.865	1.031	1.355	1.731	.646	.924	1.101	1.447	1.849	.673	.962	1.146	1.506	1.924	.725	1.037	1.236	1.625	2.075
18	.604	.863	1.029	1.352	1.727	.643	.920	1.096	1.441	1.840	.669	.956	1.140	1.498	1.913	.720	1.029	1.227	1.612	2.059
19	.602	.861	1.026	1.349	1.723	.641	.916	1.092	1.435	1.833	.666	.952	1.134	1.490	1.904	.715	1.022	1.218	1.601	2.045
20	.601	.859	1.024	1.346	1.719	.638	.913	1.088	1.430	1.826	.662	.927	1.129	1.483	1.895	.710	1.016	1.210	1.590	2.032
25	.596	.852	1.015	1.334	1.705	.629	.899	1.072	1.408	1.799	.650	.929	1.107	1.455	1.859	.692	.989	1.178	1.549	1.978
30	.592	.847	1.009	1.326	1.694	.622	.890	1.060	1.393	1.780	.641	.917	1.092	1.435	1.833	.678	.970	1.156	1.519	1.940
40	.587	.840	1.001	1.315	1.680	.613	.876	1.044	1.372	1.753	.629	.899	1.071	1.408	1.799	.660	.944	1.125	1.479	1.889
50	.584	.835	.995	1.308	1.671	.607	.867	1.034	1.358	1.735	.621	.888	1.058	1.390	1.775	.648	.927	1.105	1.452	1.855
75	.579	.828	.986	1.296	1.656	.597	.854	1.017	1.337	1.708	.608	.870	1.036	1.362	1.740	.630	.901	1.073	1.411	1.802
100	.576	.824	.982	1.290	1.648	.592	.846	1.008	1.325	1.692	.601	.860	1.024	1.346	1.719	.620	.886	1.056	1.387	1.772
125	.574	.821	.978	1.286	1.642	.588	.841	1.002	1.316	1.682	.596	.853	1.016	1.335	1.706	.613	.876	1.044	1.372	1.753
∞	.559	.799	.952	1.251	1.598	.559	.799	.952	1.251	1.598	.559	.799	.952	1.251	1.598	.559	.799	.952	1.251	1.598

TABLE 12.8. TOLERANCE FACTORS FOR NORMAL DISTRIBUTION

Factor k such that the probability is p that at least a proportion P of the distribution will be included between $\bar{\bar{x}} \pm k_3\bar{R}$ where $\bar{\bar{x}}$ is the grand mean and $\bar{R}$ is the mean range in N samples of size 5

	$p = 0.75$					$p = 0.90$					$p = 0.95$					$p = 0.99$				
N \ P	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999	0.75	0.90	0.95	0.99	0.999
4	.578	.826	.984	1.293	1.650	.651	.930	1.108	1.455	1.858	.699	.999	1.190	1.563	1.996	.800	1.143	1.362	1.789	2.284
5	.567	.811	.966	1.269	1.621	.631	.901	1.074	1.411	1.802	.673	.961	1.145	1.505	1.921	.759	1.085	1.293	1.698	2.168
6	.560	.800	.953	1.252	1.599	.616	.881	1.050	1.379	1.761	.653	.934	1.113	1.463	1.868	.730	1.044	1.244	1.634	2.087
7	.554	.792	.943	1.240	1.584	.605	.866	1.032	1.356	1.732	.639	.914	1.089	1.431	1.829	.708	1.013	1.207	1.586	2.026
8	.549	.785	.936	1.230	1.571	.597	.854	1.017	1.337	1.708	.628	.898	1.070	1.406	1.797	.691	.989	1.178	1.548	1.978
9	.545	.780	.929	1.221	1.560	.590	.844	1.005	1.321	1.688	.619	.885	1.055	1.386	1.771	.678	.969	1.155	1.518	1.939
10	.542	.776	.924	1.215	1.551	.584	.836	.996	1.309	1.672	.611	.874	1.042	1.369	1.749	.666	.953	1.135	1.492	1.906
11	.540	.772	.920	1.209	1.544	.580	.829	.987	1.298	1.658	.605	.865	1.031	1.355	1.731	.657	.939	1.119	1.471	1.879
12	.538	.769	.916	1.203	1.538	.575	.823	.980	1.288	1.646	.600	.857	1.022	1.343	1.715	.649	.927	1.105	1.452	1.855
13	.536	.766	.913	1.199	1.532	.572	.817	.974	1.280	1.635	.595	.851	1.013	1.332	1.702	.641	.917	1.093	1.436	1.835
14	.534	.763	.910	1.195	1.527	.568	.813	.969	1.273	1.626	.591	.844	1.006	1.322	1.689	.635	.908	1.082	1.422	1.816
15	.532	.761	.907	1.192	1.523	.566	.809	.964	1.267	1.618	.587	.839	1.000	1.314	1.679	.629	.900	1.073	1.410	1.801
16	.531	.759	.905	1.189	1.519	.563	.805	.959	1.261	1.611	.584	.834	.994	1.307	1.669	.624	.893	1.064	1.398	1.786
17	.530	.757	.903	1.186	1.515	.561	.802	.955	1.256	1.604	.581	.830	.989	1.300	1.660	.620	.886	1.056	1.388	1.773
18	.529	.756	.901	1.184	1.512	.559	.799	.952	1.251	1.598	.578	.826	.984	1.294	1.652	.616	.880	1.049	1.379	1.761
19	.528	.754	.899	1.181	1.509	.557	.796	.949	1.247	1.593	.575	.823	.980	1.288	1.645	.612	.875	1.043	1.371	1.751
20	.527	.753	.897	1.179	1.506	.555	.794	.946	1.243	1.587	.573	.819	.976	1.283	1.639	.609	.870	1.037	1.363	1.741
25	.523	.748	.891	1.171	1.496	.548	.783	.933	1.227	1.567	.564	.806	.960	1.262	1.612	.595	.851	1.013	1.332	1.702
30	.520	.744	.886	1.165	1.488	.543	.776	.925	1.215	1.552	.557	.796	.949	1.247	1.593	.585	.836	.996	1.309	1.573
40	.516	.738	.880	1.156	1.477	.536	.766	.913	1.199	1.532	.548	.783	.933	1.226	1.567	.571	.817	.973	1.279	1.534
50	.514	.735	.876	1.151	1.471	.531	.759	.905	1.189	1.519	.542	.774	.923	1.213	1.549	.562	.804	.958	1.259	1.509
75	.510	.729	.869	1.142	1.459	.524	.749	.893	1.173	1.499	.532	.761	.907	1.192	1.523	.549	.785	.935	1.229	1.470
100	.508	.726	.865	1.137	1.453	.520	.743	.885	1.164	1.486	.527	.753	.898	1.180	1.507	.541	.774	.922	1.211	1.448
∞	.495	.707	.843	1.107	1.415	.495	.707	.843	1.107	1.415	.495	.707	.843	1.107	1.415	.495	.707	.843	1.107	1.415

13. DISTRIBUTION OF RANGE

13.1 Moment Constants of the Mean Deviation and Range

a. Introduction

Let $x_1, x_2, \ldots, x_n$ denote a random sample of n observations and $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ denote the same observations in the ascending order of magnitude. The mean deviation m and the sample range R are defined by

$$m = \frac{1}{n} \sum_{i=1}^{n} |x_i - \bar{x}| \text{ and } R = x_{(n)} - x_{(1)}.$$

When the population sampled is normal with standard deviation σ, Table 13.1 gives the expected value, standard deviation and β_1 and β_2 for the distribution of m/σ. This table also gives the expected value (d_2), standard deviation (d_3), variance (d_3^2), β_1 and β_2, d_2/d_3^2 and d_2^2/d_3^2 for the distribution of the standardised range R/σ.

b. Application

Table 13.1 is useful for estimating σ by mean deviation or range mostly in quality control work. For example since $E(R/\sigma) = d_2$, R/d_2 is an unbiased estimate of σ and the standard error of this estimate is $\sigma d_3/d_2$. Similarly an unbiased estimate of σ can be obtained by dividing the mean deviation by its expected value.

The following table gives the standard errors of different unbiased estimators of σ, based on sample standard deviation, mean deviation and sample range.

STANDARD ERRORS OF DIFFERENT UNBIASED ESTIMATORS OF σ
(Expressed in terms of σ as unit)

Sample size (n)	S.D. estimate	M.D. estimate	Range estimate	Range estimate / S.D. estimate
2	0.756	0.756	0.756	1.00
3	0.523	0.525	0.525	1.00
4	0.422	0.430	0.427	1.01
5	0.363	0.373	0.372	1.02
6	0.323	0.334	0.335	1.04
7	0.294	0.306	0.308	1.05
8	0.272	0.283	0.288	1.06
9	0.254	0.265	0.272	1.07
10	0.239	0.250	0.259	1.08
12	0.215	0.227	0.239	1.11
15	0.191	0.201	0.218	1.14
20	0.163	0.173	0.195	1.20

It is seen that up to $n = 10$, there is very little to choose between mean deviation and range. Beyond this, relative accuracy of the range estimator falls off progressively. It is customary to estimate σ from the mean range of the observations in a number of small groups. If k samples of r observations each are available and we write the mean value of their ranges as $\bar{R}$, then $\bar{R}/d_2$ is an unbiased estimate of σ. The use of factors d_2/d_3^2 and d_2^2/d_3^2 is discussed in section 13.3.

c. Example

Twenty samples of size 5 were taken of a particular component and diameters were measured. The mean range $\bar{R}$ was 0.01435, find an estimate of σ.

From Table 13.1, the value of d_2 for $n = 5$ is 2.326 (correct to 3 decimal places). Hence an estimate of σ is

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.01435}{2.326} = 0.006169.$$

The standard error of the estimate is estimated by

$$\hat{\sigma}d_3/d_2\sqrt{20} = \bar{R}d_3/d_2^2\sqrt{20} = (0.006169)(0.8641)/(4.472136)(2.326)$$
$$= 0.0005.$$

TABLE 13.1. MOMENT CONSTANTS OF THE MEAN DEVIATION AND OF THE RANGE

n	Mean deviation					Range						
	Expectation	S.D.	variance	β_1	β_2	Expectation $= d_2$	S.D. $= d_3$	variance $= d_3^2$	β_1	β_2	d_2/d_3^2	d_2^2/d_3^2
2	0.564 190	0.4263	0.18169	0.991	3.869	1.12838	0.8525	0.72676	0.9906	3.869	1.55	1.75
3	.651 470	.3419	.11692	.417	3.286	1.69257	.8884	.78922	.4174	3.286	2.14	3.63
4	.690 988	.2970	.08822	.298	3.252	2.05875	.8798	.77407	.2735	3.188	2.66	5.48
5	0.713 650	0.2663	0.07094	0.230	3.197	2.32593	0.8641	0.74661	0.2174	3.169	3.12	7.25
6	.728 366	.2436	.05934	.187	3.161	2.53441	.8480	.71916	.1892	3.168	3.52	8.93
7	.738 698	.2258	.05101	.157	3.136	2.70436	.8332	.69424	.1742	3.174	3.90	10.53
8	.746 353	.2115	.04473	.136	3.118	2.84720	.8198	.67213	.1657	3.184	4.24	12.06
9	.752 253	.1996	.03982	.119	3.104	2.97003	.8078	.65262	.1608	3.191	4.55	13.52
10	0.756 940	0.1894	0.03589	0.106	3.0927	3.07751	0.7971	0.63531	0.1580	3.200	4.84	14.91
11	.760 753	.1807	.03266	.0961	3.0838	3.17287	.7873	.61984	.1564	3.205	5.12	16.2
12	.763 916	.1731	.02997	.0876	3.0765	3.25846	.7785	.60601	.1560	3.213	5.38	17.5
13	.766 583	.1664	.02769	.0805	3.0703	3.33598	.7704	.59353	.1559	3.220	5.62	18.8
14	.768 861	.1604	.02573	.0744	3.0650	3.40676	.7630	.58217	.1561	3.225	5.85	19.9
15	0.770 830	0.1550	0.02403	0.0692	3.0605	3.47183	0.7562	0.57186	0.1568	3.231	6.07	21.1
16	.772 548	.1501	.02254	.0647	3.0566	3.53198	.7499	.56237	.1576	3.237	6.28	22.2
17	.774 062	.1457	.02122	.0607	3.0531	3.58788	.7441	.55363	.1588	3.242	6.48	23.3
18	.775 404	.1416	.02005	.0572	3.0501	3.64006	.7386	.54554	.1598	3.248	6.67	24.3
19	.776 604	.1378	.01900	.0541	3.0473	3.68896	.7335	.53802	.1612	3.254	6.86	25.3
20	0.777 682	0.1344	0.01806	0.0513	3.0449	3.73495	0.7287	0.53097	0.1627	3.259	7.03	26.3
30	.784 474	.1098	.01206	.0338	3.0296							
60	.791 208	.0777	.00604	.0167	3.0146							

The unit is the population standard deviation.

13.2. Percentage Points of the Distribution of the Range

a. Introduction

For the case of normal population with standard deviation σ, Table 13. 2 gives for $n = 2(1)20$, the factor $1/d_2$ and some lower and upper percentage points of the distribution of the standardized range R/σ. This table is useful in setting up a control chart for range to check on the variability of a product.

b. Example

The width of slot of terminal blocks is distributed normally with standard deviation 0.001 in. Find 2.5% probability control limit for ranges of sample size 5.

We have for $n = 5$, $d_2 = 2.326$ and the upper 2.5 percent point is 4.20. Hence the central line for the R chart is $d_2\sigma = 0.002326$ and the upper control limit is 0.0042.

TABLE 13.2. PERCENTAGE POINTS OF THE DISTRIBUTION OF THE RANGE

Size of sample n	Factor $1/d_2$	Lower percentage points						Upper percentage points					
		0.1	0.5	1.0	2.5	5.0	10.0	10.0	5.0	2.5	1.0	0.5	0.1
2	0.8862	0.00	0.01	0.02	0.04	0.09	0.18	2.33	2.77	3.17	3.64	3.97	4.65
3	.5908	0.06	0.13	0.19	0.30	0.43	0.62	2.90	3.31	3.68	4.12	4.42	5.06
4	.4857	0.20	0.34	0.43	0.59	0.76	0.98	3.24	3.63	3.98	4.40	4.69	5.31
5	.4299	0.37	0.55	0.66	0.85	1.03	1.26	3.48	3.86	4.20	4.60	4.89	5.48
6	0.3946	0.54	0.75	0.87	1.06	1.25	1.49	3.66	4.03	4.36	4.76	5.03	5.62
7	.3698	0.69	0.92	1.05	1.25	1.44	1.68	3.81	4.17	4.49	4.88	5.15	5.73
8	.3512	0.83	1.08	1.20	1.41	1.60	1.83	3.93	4.29	4.61	4.99	5.26	5.82
9	.3367	0.96	1.21	1.34	1.55	1.74	1.97	4.04	4.39	4.70	5.08	5.34	5.90
10	0.3249	1.08	1.33	1.47	1.67	1.86	2.09	4.13	4.47	4.79	5.16	5.42	5.97
11	.3152	1.20	1.45	1.58	1.78	1.97	2.20	4.21	4.55	4.86	5.23	5.49	6.04
12	.3069	1.30	1.55	1.68	1.88	2.07	2.30	4.29	4.62	4.92	5.29	5.54	6.09
13	.2998	1.39	1.64	1.77	1.97	2.16	2.39	4.35	4.68	4.99	5.35	5.60	6.14
14	.2935	1.47	1.72	1.86	2.06	2.24	2.47	4.41	4.74	5.04	5.40	5.65	6.19
15	0.2880	1.55	1.80	1.93	2.14	2.32	2.54	4.47	4.80	5.09	5.45	5.70	6.23
16	.2831	1.63	1.88	2.01	2.21	2.39	2.61	4.52	4.85	5.14	5.49	5.74	6.27
17	.2787	1.69	1.94	2.07	2.27	2.45	2.67	4.57	4.89	5.18	5.54	5.78	6.31
18	.2747	1.76	2.01	2.14	2.34	2.51	2.73	4.61	4.93	5.22	5.57	5.82	6.35
19	.2711	1.82	2.07	2.20	2.39	2.57	2.79	4.65	4.97	5.26	5.61	5.85	6.38
20	0.2677	1.87	2.12	2.25	2.45	2.62	2.84	4.69	5.01	5.30	5.65	5.89	6.41

The unit is the population standard deviation.

Estimate of σ = range (or mean range) in a sample of n observations $\times 1/d_2$.

13.3 Values Associated with the Distribution of the Average Range

a. Introduction

Suppose we have k samples each of size n from a normal population with standard deviation σ. Let $R_1, R_2, \ldots, R_k$ be the sample ranges and $\bar{R}$ their average. Patnaik (*Biometrika*, 1950, **37**) showed that $\nu(\bar{R}/d_2^*)^2/\sigma^2$ is approximately distributed as χ^2 with ν degrees of freedom where the scale factor d_2^* and the equivalent degrees of freedom ν are functions of n and k. These functions are given in Table 13.3 for $n = 1(1)15$.

b Application

The significance of Patnaik's work is then that in any analysis using s, we can replace s by the more readily computed $\bar{R}/d_2^*$. This provides shortcut tests involving the use of range or mean ranges instead of mean squares. Some of these are : analysis of variance, substitute F tests, substitute t tests etc.

c. Example

The following are data on weight of antibiotic filled in vials (in some coded units). Between the morning and the afternoon production runs, the filling machine was reset and there was some question as to whether the average level was same for both the periods. There is no reason, however, to believe that variation in weight was different for the morning and afternoon runs.

Morning run sample	Afternoon run sample	
22.0	22.5	
22.5	19.5	
22.5	22.5	
24.0	22.0	
23.5	21.0	
$\bar{x}_1 = 22.9$	$\bar{x}_2 = 21.5$	
$R_1 = 2.0$	$R_2 = 3.0,$	$\bar{R} = 2.5$

From Table 13.3, for $k = 2$, $n = 5$, we have $d_2^* = 2.4$ and $\nu = 7.5$.

$$t = \frac{\sqrt{n}\,|\bar{x}_1 - \bar{x}_2|}{\sqrt{2}(\bar{R}/d_2^*)} = \frac{\sqrt{5}\,|22.9 - 21.5|}{\sqrt{2}(2.5/2.4)} = 2.12$$

The critical value of t at 5% level of significance for 7.5 degrees of freedom (from Table 4.1) is about 2.33, indicating that the process level was same for both the runs.

d. Unequal sample sizes

In case of k samples based on unequal sample sizes n_i $(i = 1, 2, \ldots, k)$, an estimate of σ may be obtained from the mean weighted range

$$\frac{\sum_{i=1}^{k} R_i(d_2/d_3^2)}{\sum_{i=1}^{k} (d_2^2/d_3^2) + \frac{1}{2}}$$

where the factors d_2/d_3^2 and d_2^2/d_3^2 are given in Table 13.1. This quantity is approximately distributed as the root mean square estimator s for $\nu = \frac{1}{2}\sum d_2^2/d_3^2$ degrees of freedom.

TABLE 13.3. VALUES ASSOCIATED WITH THE DISTRIBUTION OF THE AVERAGE RANGE*

[$\nu(\bar{R}/d_2^*)^2/\sigma^2$] is distributed approximately as χ^2 with ν degrees of freedom; $\bar{R}$ is the average range of k subgroups, each of size (n)]

Number of samples (k)	Size of sample (n): 2		3		4		5		6		7		8	
	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*
1	1.0	1.41	2.0	1.91	2.9	2.24	3.8	2.48	4.7	2.67	5.5	2.83	6.3	2.96
2	1.9	1.28	3.8	1.81	5.7	2.15	7.5	2.40	9.2	2.60	10.8	2.77	12.3	2.91
3	2.8	1.23	5.7	1.77	8.4	2.12	11.1	2.38	13.6	2.58	16.0	2.75	18.3	2.89
4	3.7	1.21	7.5	1.75	11.2	2.11	14.7	2.37	18.1	2.57	21.3	2.74	24.4	2.88
5	4.6	1.19	9.3	1.74	13.9	2.10	18.4	2.36	22.6	2.56	26.6	2.73	30.4	2.87
6	5.5	1.18	11.1	1.73	16.6	2.09	22.0	2.35	27.1	2.56	31.8	2.73	36.4	2.87
7	6.4	1.17	12.9	1.73	19.4	2.09	25.6	2.35	31.5	2.55	37.1	2.72	42.5	2.87
8	7.2	1.17	14.8	1.72	22.1	2.08	29.3	2.35	36.0	2.55	42.4	2.72	48.5	2.87
9	8.1	1.16	16.6	1.72	24.8	2.08	32.9	2.34	40.5	2.55	47.7	2.72	54.5	2.86
10	9.0	1.16	18.4	1.72	27.6	2.08	36.5	2.34	44.9	2.55	52.9	2.72	60.6	2.86
11	9.9	1.16	20.2	1.71	30.3	2.08	40.1	2.34	49.4	2.55	58.2	2.72	66.6	2.86
12	10.8	1.15	22.0	1.71	33.0	2.07	43.7	2.34	53.9	2.55	63.5	2.72	72.7	2.85
13	11.6	1.15	23.9	1.71	35.7	2.07	47.4	2.34	58.4	2.55	68.8	2.71	78.7	2.85
14	12.5	1.15	25.7	1.71	38.5	2.07	51.0	2.34	62.8	2.54	74.0	2.71	84.7	2.85
15	13.4	1.15	27.5	1.71	41.2	2.07	54.6	2.34	67.3	2.54	79.3	2.71	90.8	2.85
d_2		1.13		1.69		2.06		2.33		2.53		2.70		2.85
c.d.	0.88		1.82		2.74		3.62		4.47		5.27		6.03	

Number of samples (k)	Size of sample (n): 9		10		11		12		13		14		15	
	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*	ν	d_2^*
1	7.0	3.08	7.7	3.18	8.4	3.27	9.0	3.35	9.6	3.42	10.2	3.49	10.8	3.55
2	13.8	3.02	15.1	3.13	16.5	3.22	17.8	3.30	19.0	3.38	20.2	3.45	21.3	3.51
3	20.5	3.01	22.6	3.11	24.6	3.21	26.5	3.29	28.4	3.37	30.2	3.43	31.9	3.50
4	27.3	3.00	30.1	3.10	32.7	3.20	35.3	3.28	37.8	3.36	40.1	3.43	42.4	3.49
5	34.0	2.99	37.5	3.10	40.9	3.19	44.1	3.28	47.1	3.35	50.1	3.42	52.9	3.49
6	40.8	2.99	45.0	3.10	49.0	3.19	52.8	3.27	56.5	3.35	60.1	3.42	63.5	3.49
7	47.5	2.99	52.4	3.10	57.1	3.19	61.6	3.27	65.9	3.35	70.0	3.42	74.0	3.48
8	54.3	2.98	59.9	3.09	65.2	3.19	70.3	3.27	75.3	3.35	80.0	3.42	84.6	3.48
9	61.0	2.98	67.3	3.09	73.3	3.18	79.1	3.27	84.6	3.35	90.0	3.42	95.1	3.48
10	67.8	2.98	74.8	3.09	81.5	3.18	87.9	3.27	94.0	3.34	99.9	3.42	105.6	3.48
11	74.6	2.98	82.3	3.09	89.6	3.18	96.6	3.27	103.4	3.34	109.9	3.41	116.2	3.48
12	81.3	2.98	89.7	3.09	97.7	3.18	105.4	3.27	112.8	3.34	119.9	3.41	126.7	3.48
13	88.1	2.98	97.2	3.09	105.8	3.18	114.1	3.27	122.2	3.34	129.9	3.41	137.3	3.48
14	94.8	2.98	104.6	3.08	113.9	3.18	122.9	3.27	131.5	3.34	139.8	3.41	147.8	3.48
15	101.6	2.98	112.1	3.08	122.1	3.18	131.7	3.26	140.9	3.34	149.8	3.41	158.3	4.48
d_2		2.97		3.08		3.17		3.26		3.34		3.41		3.47
c.d.	6.76		7.45		8.12		8.76		9.38		9.97		10.54	

* In general the degrees of freedom will be given approximately by the reciprocal of $(-2+2\sqrt{1+2(\text{c.v.})^2/k})$ where c.v. is the coefficient of variation (d_3/d_2) of the range and k is the number of subgroups. Also d_2^* is given approximately by d_2 (i.e., the infinity value of d_2^*) times $(1+1/4\nu)$. Values of ν are also very readily built up from the constant differences.

Note : c.d = constant difference.

13.4. Percentage Points of the Studentized Range

a. Introduction

The studentized range used in tests of means is defined as $q = R/s_\nu$, where s_ν^2 is an independent mean-square estimate of σ^2 based on ν degrees of freedom. Table 13.4 gives lower and upper 1% and 5% points when the population sampled is normal. The entries in the table correspond to a given number of degrees of freedom for s and the sample size for R.

b. Application

The main use of this statistic is in analysis of variance where it serves as an alternative to the F test. In the simplest case when there are k groups each of r observations, we compute the value of the statistic $q = \sqrt{r}(\bar{x}_{(k)} - \bar{x}_{(1)})/s$. Where $\bar{x}_{(k)}$ and $\bar{x}_{(1)}$ are the largest and smallest group means respectively and s^2 is the error mean square in an analysis of variance table. We get the critical value of q by reading the upper percentage point for $k(r-1)$ degrees of freedom and sample size k. However this table in conjunction with Table 13.3 provide short cut tests in analysis of variance through use of range. For this purpose, we substitute $\bar{R}/d_2^*$ for s where $\bar{R}$ is the mean of the ranges of k groups and d_2^* is an appropriate factor obtained from Table 13.3 for k samples each of size r. The equivalent degrees of freedom ν is also read from this table. The critical value of the statistic $q = \sqrt{r}(\bar{x}_{(k)} - \bar{x}_{(1)})/(\bar{R}/d_2^*)$ is obtained by reading the upper percentage point of the studentized range for ν degrees of freedom and sample size k.

c. Example

The melting point of a chemical was determined thrice on each of four thermometers:

MELTING POINT IN DEGREES CENTIGRADE

	Thermometers				
	1	2	3	4	
	174.0	173.0	171.5	173.5	
	173.0	172.0	171.0	171.0	
	173.5	173.0	173.0	172.5	
$\bar{x}$	173.5	172.67	171.83	172.33	max $\bar{x}$ = 173.5, min $\bar{x}$ = 171.83
R	1.0	1.0	2.0	2.5	$\bar{R}$ = 1.625

Do the thermometers read differently ?

We have from Table 13.3 for 4 samples and each of size 3, $d_2^* = 1.75$ and $\nu = 7.5$

$$q = \frac{\sqrt{3}(173.5 - 171.83)}{(1.625/1.75)} = 3.12$$

From Table 13.4, the upper 5% point for studentized range for sample size 4 and for 7.5 degrees of freedom (by linear interpolation) is 4.61. Hence this test does not lead to a rejection of the null hypotheses.

PERCENTAGE POINTS OF THE STUDENTIZED RANGE $q = R/s_\nu$

Lower 5% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	0.09	0.43	0.75	1.01	1.20	1.37	1.52	1.63	1.74	1.83	1.91	1.98	2.05	2.12	2.17	2.22	2.26	2.30	2.34
11	.09	.43	.75	.01	.21	.38	.52	.64	.75	.84	.92	2.00	.07	.13	.18	.24	.28	.33	.37
12	.09	.43	.75	.01	.21	.38	.53	.65	.76	.85	.93	.01	.08	.14	.20	.26	.30	.34	.38
13	.09	.43	.75	.01	.22	.39	.53	.65	.76	.86	.94	.02	.09	.15	.21	.27	.31	.36	.40
14	.09	.43	.75	.01	.22	.39	.54	.66	.77	.86	.95	.03	.10	.16	.22	.28	.32	.37	.41
15	0.09	0.43	0.75	1.01	1.22	1.39	1.54	1.66	1.77	1.87	1.95	2.03	2.11	2.17	2.23	2.29	2.34	2.38	2.43
16	.09	.43	.75	.01	.22	.39	.54	.67	.78	.87	.96	.04	.11	.18	.24	.30	.34	.39	.44
17	.09	.43	.75	.01	.22	.40	.55	.67	.78	.88	.97	.05	.12	.19	.25	.30	.35	.40	.45
18	.09	.43	.75	.02	.22	.40	.55	.67	.79	.88	.97	.05	.12	.19	.25	.31	.36	.41	.45
19	.09	.43	.75	.02	.23	.40	.55	.68	.79	.89	.98	.05	.13	.20	.26	.32	.37	.42	.46
20	0.09	0.43	0.75	1.02	1.23	1.40	1.55	1.68	1.79	1.89	1.98	2.06	2.13	2.20	2.27	2.32	2.37	2.42	2.47
24	.09	.43	.75	.02	.23	.41	.56	.69	.80	.90	.99	.08	.15	.22	.28	.34	.39	.45	.49
30	.09	.43	.76	.02	.24	.41	.57	.70	.81	.92	2.01	.09	.17	.24	.30	.36	.41	.47	.52
40	.09	.43	.76	.02	.24	.42	.57	.71	.82	.93	.02	.10	.18	.26	.32	.38	.43	.49	.54
60	0.09	0.43	0.76	1.02	1.24	1.43	1.58	1.72	1.83	1.94	2.04	2.12	2.20	2.28	2.34	2.40	2.46	2.52	2.57
120	.09	.43	.76	.03	.25	.43	.59	.73	.85	.96	.06	.14	.22	.30	.36	.43	.49	.54	.60
∞	0.09	0.43	0.76	1.03	1.25	1.44	1.60	1.74	1.86	1.97	2.07	2.16	2.24	2.32	2.39	2.45	2.52	2.57	2.62

Upper 5% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	18.0	27.0	32.8	37.1	40.4	43.1	45.4	47.4	49.1	50.6	52.0	53.2	54.3	55.4	56.3	57.2	58.0	58.8	59.6
2	6.09	8.3	9.8	10.9	11.7	12.4	13.0	13.5	14.0	14.4	14.7	15.1	15.4	15.7	15.9	16.1	16.4	16.6	16.8
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46	9.72	9.95	10.15	10.35	10.52	10.69	10.84	10.98	11.11	11.24
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83	8.03	8.21	8.37	8.52	8.66	8.79	8.91	9.03	9.13	9.23
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	3.46	4.34	4.90	5.31	5.63	5.89	6.12	6.32	6.49	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.09	7.17
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	3.20	3.95	4.42	4.76	5.02	5.24	5.43	5.60	5.74	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83	5.93	6.03	6.11	6.20	6.27	6.34	6.40	6.47
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49	5.61	5.71	5.81	5.90	5.99	6.06	6.14	6.20	6.26	6.33
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.40	5.51	5.62	5.71	5.80	5.88	5.95	6.03	6.09	6.15	6.21
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53	5.63	5.71	5.79	5.86	5.93	6.00	3.05	6.11
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46	5.55	5.64	5.72	5.79	5.85	5.92	5.97	6.03
15	3.01	3.67	4.08	4.37	4.60	4.78	4.94	5.08	5.20	5.31	5.40	5.49	5.58	5.65	5.72	5.79	5.85	5.90	5.96
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35	5.44	5.52	5.59	5.66	5.72	5.79	5.84	5.90
17	2.98	3.63	4.02	4.30	4.52	4.71	4.86	4.99	5.11	5.21	5.31	5.39	5.47	5.55	5.61	5.68	5.74	5.79	5.84
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23	5.32	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.50	5.54	5.59
30	2.89	3.49	3.84	4.10	4.30	4.46	4.60	4.72	4.83	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.48
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.74	4.82	4.91	4.98	5.05	5.11	5.16	5.22	5.27	5.31	5.36
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.16	5.20	5.24
120	2.80	3.36	3.69	3.92	4.10	4.24	4.36	4.48	4.56	4.64	4.72	4.78	4.84	4.90	4.95	5.00	5.05	5.09	5.13
∞	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62	4.68	4.74	4.80	4.85	4.89	4.93	4.97	5.01

n is the size of sample from which the range is obtained and ν is the number of degrees of freedom of s_ν.

Lower 1% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
10	0.02	0.18	0.42	0.64	0.81	0.96	1.11	1.23	1.34	1.41	1.50	1.57	1.62	1.70	1.74	1.81	1.84	1.88	1.92
11	.02	.18	.42	.64	.82	.97	.12	.24	.35	.43	.52	.58	.64	.71	.76	.82	.86	.91	.94
12	.02	.18	.42	.64	.82	.98	.12	.24	.35	.44	.53	.60	.65	.73	.77	.84	.88	.92	.96
13	.02	.18	.42	.64	.83	.98	.13	.25	.36	.45	.54	.61	.66	.74	.79	.85	.89	.94	.98
14	.02	.18	.42	.65	.83	.99	.13	.25	.37	.46	.55	.62	.68	.76	.80	.87	.91	.95	2.00
15	0.02	0.18	0.42	0.65	0.83	0.99	1.14	1.26	1.37	1.46	1.55	1.63	1.69	1.76	1.81	1.88	1.92	1.97	2.01
16	.02	.18	.42	.65	.83	.99	.14	.26	.37	.47	.56	.63	.70	.77	.82	.89	.93	.98	.02
17	.02	.18	.42	.65	.84	1.00	.14	.27	.38	.48	.57	.64	.70	.78	.83	.90	.94	.99	.04
18	.02	.18	.42	.65	.84	.00	.15	.27	.38	.48	.57	.65	.71	.79	.84	.91	.95	2.00	.05
19	.02	.18	.43	.65	.84	.00	.15	.28	.39	.48	.58	.65	.72	.80	.85	.91	.96	.01	.06
20	0.02	0.18	0.43	0.65	0.84	1.01	1.15	1.28	1.39	1.49	1.58	1.66	1.72	1.80	1.85	1.92	1.97	2.01	2.06
24	.02	.18	.43	.65	.85	.01	.16	.29	.40	.50	.60	.67	.74	.82	.88	.94	.99	.05	.09
30	.02	.18	.43	.66	.85	.02	.17	.30	.41	.52	.61	.69	.76	.84	.90	.97	2.02	.07	.12
40	.02	.18	.43	.66	.85	.02	.18	.31	.43	.53	.63	.71	.79	.86	.92	.99	.04	.10	.15
60	0.02	0.18	0.43	0.66	0.86	1.03	1.19	1.32	1.44	1.55	1.64	1.73	1.81	1.88	1.95	2.02	2.07	2.13	2.18
120	.02	.18	.43	.66	.86	.04	.20	.33	.45	.56	.66	.75	.83	.91	.98	.04	.10	.16	.21
∞	0.02	0.19	0.43	0.66	0.87	1.05	1.20	1.34	1.47	1.58	1.68	1.77	1.86	1.93	2.01	2.08	2.14	2.20	2.25

Upper 1% points

ν \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	90.0	135.0	164.0	186.0	202.0	216.0	227.0	237.0	246.0	253.0	260.0	266.0	272.0	277.0	282.0	286.0	290.0	294.0	298.0
2	14.0	19.0	22.3	24.7	26.6	28.2	29.5	30.7	31.7	32.6	33.4	34.1	34.8	35.4	46.0	36.5	37.0	37.5	37.9
3	8.26	10.6	12.2	13.3	14.2	15.0	15.6	16.2	16.7	17.1	17.5	17.9	18.2	18.5	18.8	19.1	19.3	19.5	19.8
4	6.51	8.12	9.17	9.96	10.6	11.1	11.5	11.9	12.3	12.6	12.8	13.1	13.3	13.5	13.7	13.9	14.1	14.2	14.4
5	5.70	6.97	7.80	8.42	8.91	9.32	9.67	9.97	10.24	10.48	10.70	10.89	11.08	11.24	11.40	11.55	11.68	11.81	11.93
6	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.49	9.65	9.81	9.95	10.08	10.21	10.32	10.43	10.54
7	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71	8.86	9.00	9.12	9.24	9.35	9.46	9.55	9.65
8	4.74	5.63	6.20	6.63	6.96	7.24	7.47	7.68	7.87	8.03	8.18	8.31	8.44	8.55	8.66	8.76	8.85	8.94	9.03
9	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.32	7.49	7.65	7.78	7.91	8.03	8.13	8.23	8.32	8.41	8.49	8.57
10	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.48	7.60	7.71	7.81	7.91	7.99	8.07	8.15	8.22
11	4.39	5.14	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.25	7.36	7.46	7.56	7.65	7.73	7.81	7.88	7.95
12	4.32	5.04	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06	7.17	7.26	7.36	7.44	7.52	7.59	7.66	7.73
13	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90	7.01	7.10	7.19	7.27	7.34	7.42	7.48	7.55
14	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77	6.87	6.96	7.05	7.12	7.20	7.27	7.33	7.39
15	4.17	4.83	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66	6.76	6.84	6.93	7.00	7.07	7.14	7.20	7.26
16	4.13	4.78	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56	6.66	6.74	6.82	6.90	6.97	7.03	7.09	7.15
17	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48	6.57	6.66	6.73	6.80	6.87	6.94	7.00	7.05
18	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41	6.50	6.58	6.65	6.72	6.79	6.85	6.91	6.96
19	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14	6.25	6.34	6.43	6.51	6.58	6.65	6.72	7.78	6.84	6.89
20	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.29	6.37	6.45	6.52	6.59	6.65	6.71	6.76	6.82
24	3.96	4.54	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11	6.19	6.26	6.33	6.39	6.45	6.51	6.56	6.61
30	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76	5.85	5.93	6.01	6.08	6.14	6.20	6.26	6.31	6.36	6.41
40	3.82	4.37	4.70	4.93	5.11	5.27	5.39	5.50	5.60	5.69	5.77	5.84	5.90	5.96	6.02	6.07	6.12	6.17	6.21
60	3.76	4.28	4.60	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60	5.67	5.73	5.79	5.84	5.89	5.93	5.98	6.02
120	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.38	5.44	5.51	5.56	5.61	5.66	5.71	5.75	5.79	5.83
∞	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29	5.35	5.40	5.45	5.49	5.54	5.57	5.61	5.65

n is the size of the sample from which the range is obtained and ν is the number of degrees of freedom of s_ν.

a. Lagrange's formula

Given the values of a function $f(x)$ at $x = x_i$ $(i = 1, 2, \ldots, m)$, the interpolated value at any value of x is given by formula

$$f(x) = \sum_{i=1}^{m} A_i(x) f(x_i)$$

where
$$A_i(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_m)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_m)}$$

This formula due to Lagrange gives directly the equation to the $(m-1)$-th degree polynomial which coincides with $f(x)$ at the chosen points.

The coefficients $A_i(x)$ are tabulated in Tables 14.1 to 14.4 for the special case where the chosen arguments x_i are at equal intervals and for $m = 3, 4, 5$ and 6. These tables will be found very useful for polynomial interpolation since they avoid the computation of a table of differences (see chapter VI of Part I).

b. Application

Suppose a function $f(x)$ is tabulated at intervals of 10, say at $x = 30, 40, 50, 60, 70, 80, \ldots$, and the value of the function is required at 52. Let us decide on a four point interpolation formula ($m = 4$) and choose the arguments 40, 50, 60, 70. In Table 14.2 the four arguments are always written as $-1, 0, 1, 2$, so that a suitable translation and scale transformation is required to apply the formula. In the present case the origin is 50 and the scale is 10, the width of the interval of tabulation. Now we compute $u = (52-50)/10 = 0.20$, subtracting the value of the origin and dividing by the width of interval. Reading from Table 14.2 we find the values of $A_{-1}, A_0, A_1\ A_2$ corresponding to $u = 0.20$. Then the interpolated value for $f(52)$ is $A_{-1} f(40) + A_0 f(50) + A_1 f(60) + A_2 f(70)$. We could have chosen any set of four consecutive arguments. But it is better, if possible, to choose the arguments symmetrically about the interval containing 52.

Example. The following are the 1% values of chi-square for different degrees of freedom.

d.f.	χ^2
30	14.95
40	22.16
50	29.71
60	37.49
70	45.44
80	53.54
90	61.76
100	70.07

Find by interpolation the values of χ^2 for (i) 52 d.f. and (ii) 33 d.f.

(I) EVALUATION OF THE 1% VALUE OF χ^2 FOR 52 D.F. USING 4-POINT INTERPOLATION (TABLE 14.2)

$$u = \frac{52-50}{10} = 0.20$$

argument x	$f(x)$	coefficients for $u = 0.20$	col. (2) × col. (3)
(1)	(2)	(3)	(4)
40	22.16	$A_{-1} = -0.048$	−1.0637
50	29.71	$A_0 = 0.864$	25.6694
60	37.49	$A_1 = 0.216$	8.0978
70	45.44	$A_1 = -0.032$	−1.4541
total	—	1.000	31.2494 (Required value)

(II) EVALUATION OF THE 1% VALUE OF χ^2 FOR 33 D.F. USING 4-POINT INTERPOLATION (TABLE 14.2)

$$u = \frac{33-40}{10} = -0.70$$

argument x	$f(x)$	coefficient for $u = -0.70$	col. (2) × col. (3)
(1)	(2)	(3)	(4)
30	14.95	$A_{-1} = 0.5355$	8.0057
40	22.16	$A_0 = 0.6885$	15.2572
50	29.71	$A_1 = -0.2835$	−8.4228
60	37.49	$A_2 = 0.0595$	2.2307
total	—	1.0000	17.0708 (Required value)

In this case it is not possible to choose tabular values symmetrically on either side of x. The four tabular arguments closest to x are 30, 40, 50 and 60.

c. Another table

1. NATIONAL BUREAU OF STANDARDS (1944): *Tables of Lagrangian Interpolation Coefficients*, Columbia University Press.

Coverage: Formula	Coefficients given to	u
3 pt.	9 dec.	−1(0.0001)1
4 pt.	10 dec.	−1(0.001) 0(0.0001) 1 (0.001) 2
5 pt.	10 dec.	−2(0.001) 2
6 pt.	10 dec.	−2(0.01)0 (0.001)1 (0.01)3
7 pt.	10 dec.	−3(0.1)−1 (0.001)1 (0.1)3
8 pt.	10 dec.	−3(0.1)0 (0.001)1 (0.1)4
9 pt.	10 dec.	−4(0.1)4
10 pt.	10 dec.	−4(0.1)5
11 pt.	10 dec.	−5(0.1)5

TABLE 14.1. THE LAGRANGIAN INTERPOLATION COEFFICIENTS

Three-point formula (Quadratic)

u	A_{-1}	A_0	A_1	u	A_{-1}	A_0	A_1
.00	−.00000	1.00000	.00000	.50	−.12500	.75000	.37500
.01	−.00495	.99990	.00505	.51	−.12495	.73990	.38505
.02	−.00980	.99960	.01020	.52	−.12480	.72960	.39520
.03	−.01455	.99910	.01545	.53	−.12455	.71910	.40545
.04	−.01920	.99840	.02080	.54	−.12420	.70840	.41580
.05	−.02375	.99750	.02625	.55	−.12375	.69750	.42625
.06	−.02820	.99640	.03180	.56	−.12320	.68640	.43680
.07	−.03255	.99510	.03745	.57	−.12255	.67510	.44745
.08	−.03680	.99360	.04320	.58	−.12180	.66360	.45820
.09	−.04095	.99190	.04905	.59	−.12095	.65190	.46905
.10	−.04500	.99000	.05500	.60	−.12000	.64000	.48000
.11	−.04895	.98790	.06105	.61	−.11895	62790	.49105
.12	−.05280	.98560	.06720	.62	−.11780	61560	.50220
.13	−.05655	.98310	.07345	.63	−.11655	60310	.51345
.14	−.06020	.98040	.07980	.64	−.11520	.59040	.52480
.15	−.06375	.97750	.08625	.65	−.11375	.57750	.53625
.16	−.06720	.97440	.09280	.66	−.11220	.56440	.54780
.17	−.07055	.97110	.09945	.67	−.11055	.55110	.55945
.18	−.07380	.96760	.10620	.68	−.10880	.53760	.57120
.19	−.07695	.96390	.11305	.69	−.10695	.52390	.58305
.20	−.08000	.96000	.12000	.70	−.10500	.51000	.59500
.21	−.08295	.95590	.12705	.71	−.10295	.49590	.60705
.22	−.08580	.95160	.13420	.72	−.10080	.48160	.61920
.23	−.08855	.94710	.14145	.73	−.09855	.46710	.63145
.24	−.09120	.94240	.14880	.74	−.09620	.45240	.64380
.25	−.09375	.93750	.15625	.75	−.09375	.43750	.65625
.26	−.09620	.93240	.16380	.76	−.09120	.42240	.66880
.27	−.09855	.92710	.17145	.77	−.08855	.40710	.68145
.28	−.10080	.92160	.17920	.78	−.08580	.39160	.69420
.29	−.10295	.91590	.18705	.79	−.08295	.37590	.70705
.30	−.10500	.91000	.19500	.80	−.08000	.36000	.72000
.31	−.10695	.90390	.20305	.81	−.07695	.34390	.73305
.32	−.10880	.89760	.21120	.82	−.07380	.32760	.74620
.33	−.11055	.89110	.21945	.83	−.07055	.31110	.75945
.34	−.11220	.88440	.22780	.84	−.06720	.29440	77280
.35	−.11375	.87750	.23625	.85	−.06375	.27750	.78625
.36	−.11520	.87040	.24480	.86	−.06020	.26040	.79980
.37	−.11655	86310	.25345	.87	−.05655	.24310	.81345
.38	−.11780	.85560	.26220	.88	−.05280	.22560	.82720
.39	−.11895	.84790	.27105	.89	−.04895	.20790	.84105
.40	−.12000	.84000	.28000	.90	−.04500	.19000	.85500
.41	−.12095	.83190	.28905	.91	−.04095	.17190	.86905
.42	−.12180	.82360	.29820	.92	−.03680	.15360	.88320
.43	−.12255	.81510	.30745	.93	−.03255	.13510	.89745
.44	−.12320	80640	.31680	.94	−.02820	.11640	.91180
.45	−.12375	.79750	.32625	.95	−.02375	.09750	.92625
.46	−.12420	.78840	.33580	.96	−.01920	.07840	.94080
.47	−.12455	.77910	.34545	.97	−.01455	.05910	.95545
.48	−.12480	.76960	.35520	.98	−.00980	.03960	.97020
.49	−.12495	.75990	.36505	.99	−.00495	.01990	.98505

Note: If the arguments chosen are $x_1 < x_2 < x_3$ and interpolation is required at $x (x_2 < x < x_3)$, compute $u = (x - x_2)/h$, where $h = x_2 - x_1 = x_3 - x_2$. Read the three entries A_{-1}, A_0, A_1 corresponding to u. Then the interpolated value is $A_{-1} f(x_1) + A_0\, f(x_2) + A_1\, f(x_3)$. It x is such that $x_1 < x < x_2$, then compute $v = (x_2 - x)/h$ and use the formula $A_{-1} f(x_3) + A_0 f(x_2) + A_1 f(x_1)$.

TABLE 14.2. THE LAGRANGIAN INTERPOLATION COEFFICIENTS

Four-point formula (Cubic)

u	A_{-1}	A_0	A_1	A_2	
.01	−.0032835	.9949005	.0100495	−.0016665	.99
.02	−.0064680	.9896040	.0201960	−.0033320	.98
.03	−.0095545	.9841135	.0304365	−.0049955	.97
.04	−.0125440	.9784320	.0407680	−.0066560	.96
.05	−.0154375	.9725625	.0511875	−.0083125	.95
.06	−.0182360	.9665080	.0616920	−.0099640	.94
.07	−.0209405	.9602715	.0722785	−.0116095	.93
.08	−.0235520	.9538560	.0829440	−.0132480	.92
.09	−.0260715	.9472645	.0936855	−.0148785	.91
.10	−.0285000	.9405000	.1045000	−.0165000	.90
.11	−.0308385	.9335655	.1153845	−.0181115	.89
.12	−.0330880	.9264640	.1263360	−.0197120	.88
.13	−.0352495	.9191985	.1373515	−.0213005	.87
.14	−.0373240	.9117720	.1484280	−.0228760	.86
.15	−.0393125	.9041875	.1595625	−.0244375	.85
.16	−.0412160	.8964480	.1707520	−.0259840	.84
.17	−.0430355	.8885565	.1819935	−.0275145	.83
.18	−.0447720	.8805160	.1932840	−.0290280	.82
.19	−.0464265	.8723295	.2046205	−.0305235	.81
.20	−.0480000	.8640000	.2160000	−.0320000	.80
.21	−.0494935	.8555305	.2274195	−.0334565	.79
.22	−.0509080	.8469240	.2388760	−.0348920	.78
.23	−.0522445	.8381835	.2503665	−.0363055	.77
.24	−.0535040	.8293120	.2618880	−.0376960	.76
.25	−.0546875	.8203125	.2734375	−.0390625	.75
.26	−.0557960	.8111880	.2850120	−.0404040	.74
.27	−.0568305	.8019415	.2966085	−.0417195	.73
.28	−.0577920	.7925760	.3082240	−.0430080	.72
.29	−.0586815	.7830945	.3198555	−.0442685	.71
.30	−.0595000	.7735000	.3315000	−.0455000	.70
	A_2	A_1	A_0	A_{-1}	u

u	A_{-1}	A_0	A_1	A_2	
.31	−.0602485	.7637955	.3431545	−.0467015	.69
.32	−.0609280	.7539840	.3548160	−.0478720	.68
.33	−.0615395	.7440685	.3664815	−.0490105	.67
.34	−.0620840	.7340520	.3781480	−.0501160	.66
.35	−.0625625	.7239375	.3898125	−.0511875	.65
.36	−.0629760	.7137280	.4014720	−.0522240	.64
.37	−.0633255	.7034265	.4131235	−.0532245	.63
.38	−.0636120	.6930360	.4247640	−.0541880	.62
.39	−.0638365	.6825595	.4363905	−.0551135	.61
.40	−.0640000	.6720000	.4480000	−.0560000	.60
.41	−.0641035	.6613605	.4595895	−.0568465	.59
.42	−.0641480	.6506440	.4711560	−.0576520	.58
.43	−.0641345	.6398535	.4826965	−.0584155	.57
.44	−.0640640	.6289920	.4942080	−.0591360	.56
.45	−.0639375	.6180625	.5056875	−.0598125	.55
.46	−.0637560	.6070680	.5171320	−.0604440	.54
.47	−.0635205	.5960115	.5285385	−.0610295	.53
.48	−.0632320	.5848960	.5399040	−.0615680	.52
.49	−.0628915	.5737245	.5512255	−.0620585	.51
.50	−.0625000	.5625000	.5625000	−.0625000	.50
	A_2	A_1	A_0	A_{-1}	u

u	A_1	A_0	A_1	A_2	
1.10	.0165000	−.0945000	1.0395000	.0385000	−.10
1.20	.0320000	−.1760000	1.0560000	.0880000	−.20
1.30	.0455000	−.2415000	1.0465000	.1495000	−.30
1.40	.0560000	−.2880000	1.0080000	.2240000	−.40
1.50	.0625000	−.3125000	.9375000	.3125000	−.50
1.60	.0640000	−.3120000	.8320000	.4160000	−.60
1.70	.0595000	−.2835000	.6885000	.5355000	−.70
1.80	.0480000	−.2240000	.5040000	.6720000	−.80
1.90	.0285000	−.1305000	.2755000	.8265000	−.90
	A_2	A_1	A_0	A_{-1}	u

Note For values of u in the right hand side column of the tables the coefficients are to be read as indicated in the bottom row of the tables. Thus for u = .74, A_{-1} = −.0404040, A_0 = .2850120 A_1 = .8111880, A_2 = −.0557960.

TABLE 14.3. THE LAGRANGIAN INTERPOLATION COEFFICIENTS

Five-point formula (Quadric)

u	A_{-2}	A_{-1}	A_0	A_1	A_2
0.02	0.0016493	−0.0130654	0.9995000	0.0135986	−0.0016827
0.04	0.0032614	−0.0255898	0.9980006	0.0277222	−0.0033946
0.06	0.0048325	−0.0375662	0.9955032	0.0423618	−0.0051315
0.08	0.0063590	−0.0489882	0.9920102	0.0575078	−0.0068890
0.10	0.0078375	−0.0598500	0.9875250	0.0731500	−0.0086625
0.12	0.0092646	−0.0701466	0.9820518	0.0892774	−0.0104474
0.14	0.0106373	−0.0798734	0.9755960	0.1058786	−0.0122387
0.16	0.0119526	−0.0890266	0.9681638	0.1229414	−0.0140314
0.18	0.0132077	−0.0976030	0.9597624	0.1404530	−0.0158203
0.20	0.0144000	−0.1056000	0.9504000	0.1584000	−0.0176000
0.22	0.0155269	−0.1130158	0.9400856	0.1767682	−0.0193651
0.24	0.0165862	−0.1198490	0.9288294	0.1955430	−0.0211098
0.26	0.0175757	−0.1260990	0.9166424	0.2147090	−0.0228283
0.28	0.0184934	−0.1317658	0.9035366	0.2342502	−0.0245146
0.30	0.0193375	−0.1368500	0.8895250	0.2541500	−0.0261625
0.32	0.0201062	−0.1413530	0.8746214	0.2743910	−0.0277658
0.34	0.0207981	−0.1452766	0.8588408	0.2949554	−0.0293179
0.36	0.0214118	−0.1486234	0.8421990	0.3158246	−0.0308122
0.38	0.0219461	−0.1513966	0.8247128	0.3369794	−0.0322419
0.40	0.0224000	−0.1536000	0.8064000	0.3584000	−0.0336000
0.42	0.0227725	−0.1552382	0.7872792	0.3800658	−0.0348795
0.44	0.0230630	−0.1563162	0.7673702	0.4019558	−0.0360730
0.46	0.0232709	−0.1568398	0.7466936	0.4240482	−0.0371731
0.48	0.0233958	−0.1568154	0.7252710	0.4463206	−0.0381722
0.50	0.0234375	−0.1562500	0.7031250	0.4687500	−0.0390625
0.52	0.0233958	−0.1551514	0.6802790	0.4913126	−0.0398362
0.54	0.0232709	−0.1535278	0.6567576	0.5139842	−0.0404851
0.56	0.0230630	−0.1513882	0.6325862	0.5367398	−0.0410010
0.58	0.0227725	−0.1487422	0.6077912	0.5595538	−0.0413755
0.60	0.0224000	−0.1456000	0.5824000	0.5824000	−0.0416000
0.62	0.0219461	−0.1419726	0.5564408	0.6052514	−0.0416659
0.64	0.0214118	−0.1378714	0.5299430	0.6280806	−0.0415642
0.66	0.0207981	−0.1333086	0.5029368	0.6508594	−0.0412859
0.68	0.0201062	−0.1282970	0.4754534	0.6735590	−0.0408218
0.70	0.0193375	−0.1228500	0.4475250	0.6961500	−0.0401625
0.72	0.0184934	−0.1169818	0.4191846	0.7186022	−0.0392986
0.74	0.0175757	−0.1107070	0.3904664	0.7408850	−0.0382203
0.76	0.0165862	−0.1040410	0.3614054	0.7629670	−0.0369178
0.78	0.0155269	−0.0969998	0.3320376	0.7848162	−0.0353811
0.80	0.0144000	−0.0896000	0.3024000	0.8064000	−0.0336000
0.82	0.0132077	−0.0818590	0.2725304	0.8276850	−0.0315643
0.84	0.0119526	−0.0737946	0.2424678	0.8486374	−0.0292634
0.86	0.0106373	−0.0654254	0.2122520	0.8692226	−0.0266867
0.88	0.0092646	−0.0567706	0.1819238	0.8894054	−0.0238234
0.90	0.0078375	−0.0478500	0.1515250	0.9091500	−0.0206625
0.92	0.0063590	−0.0386842	0.1210982	0.9284198	−0.0171930
0.94	0.0048325	−0.0292942	0.0906872	0.9471778	−0.0134035
0.96	0.0032614	−0.0197018	0.0603366	0.9653862	−0.0092826
0.98	0.0016493	−0.0099294	0.0300920	0.9830066	−0.0048187

Note: If the arugments chosen are $x_1<x_2<x_3<x_4<x_5$ and interpolation is required at $x(x_3<x<x_4)$, compute $u=(x-x_3)/h$, where h is the interval of the argument. Read the entries A_{-2}, A_{-1}, A_0, A_1, A_2, corresponding to u. Then the interpolated value is $A_{-2}f(x_1)+A_{-1}f(x_2)+A_0f(x_3)+A_1f(x_4)+A_2f(x_5)$. If x is such that $x_2<x<x_3$, then compute $u=(x_3-x)/h$ and use the formula $A_{-2}f(x_5)+A_{-1}f(x_4)+A_0f(x_3)+A_1f(x_2)+A_2f(x_1)$.

TABLE 14.4. THE LAGRANGIAN INTERPOLATION COEFFICIENTS

Six-point formula (Quintic)

u	A_{-2}	A_{-1}	A_0	A_1	A_2	A_3	
.01	.0004957921	−.0049333767	.9965420858	.0100660817	−.0025038746	.0003332917	.99
.02	.0009830066	−.0097336932	.9928367064	.0202619736	−.0050143268	.0006663334	.98
.03	.0014614085	−.0144012590	.9888864505	.0305841170	−.0075295922	.0009988752	.97
.04	.0019307725	−.0189364224	.9846939648	.0410289152	−.0100478976	.0013306675	.96
.05	.0023908828	−.0233395703	.9802619531	.0515927344	−.0125674609	.0016614609	.95
.06	.0028415335	−.0276111276	.9755931752	.0622719048	−.0150864924	.0019910065	.94
.07	.0032825281	−.0317515567	.9706904458	.0730627217	−.0176031946	.0023190557	.93
.08	.0037136794	−.0357613568	.9655566336	.0839614464	−.0201157632	.0026453606	.92
.09	.0041348096	−.0396410640	.9601946604	.0949643071	−.0226223873	.0029696742	.91
.10	.0045457500	−.0433912500	.9546075000	.1060675000	−.0251212500	.0032917500	.90
.11	.0049463412	−.0470125223	.9487981771	.1172671904	−.0276105290	.0036113426	.89
.12	.0053364326	−.0505055232	.9427697664	.1285595136	−.0300883968	.0039282074	.88
.13	.0057158827	−.0538709296	.9365253917	.1399405758	−.0325530217	.0042421011	.87
.14	.0060845585	−.0571094524	.9300682248	.1514064552	−.0350025676	.0045527815	.86
.15	.0064423359	−.0602218359	.9234014844	.1629532031	−.0374351953	.0048600078	.85
.16	.0067890995	−.0632088576	.9165284352	.1745768448	−.0398490624	.0051635405	.84
.17	.0071247422	−.0660713273	.9094523870	.1862733805	−.0422423240	.0054631416	.83
.18	.0074491654	−.0688100868	.9021766936	.1980387864	−.0446131332	.0057585746	.82
.19	.0077622787	−.0714260096	.8947047517	.2098690158	−.0469596417	.0060496051	.81
.20	.0080640000	−.0739200000	.8870400000	.2217600000	−.0492800000	.0063360000	.80
.21	.0083542553	−.0762929929	.8791859183	.2337076492	−.0515723583	.0066175284	.79
.22	.0086329786	−.0785459532	.8711460264	.2457078536	−.0538348668	.0068939614	.78
.23	.0089001118	−.0806798752	.8629238830	.2577564845	−.0560656760	.0071650719	.77
.24	.0091556045	−.0826957824	.8545230848	.2698493952	−.0582629376	.0074306355	.76
.25	.0093994141	−.0845947266	.8459472656	.2819824219	−.0604248047	.0076904297	.75
.26	.0096315055	−.0863777876	.8372000952	.2941513848	−.0625494324	.0079442345	.74
.27	.0098518513	−.0880460729	.8282852783	.3063520892	−.0646349783	.0081918324	.73
.28	.0100604314	−.0896007168	.8192065536	.3185803264	−.0666796032	.0084330086	.72
.29	.0102572328	−.0910428802	.8099676929	.3308318746	−.0686814711	.0086675510	.71
.30	.0104422500	−.0923737500	.8005725000	.3431025000	−.0706387500	.0088952500	.70
.31	.0106154844	−.0935945385	.7910248096	.3553879579	−.0725496127	.0091158993	.69
.32	.0107769446	−.0947064832	.7813284864	.3676839936	−.0744122368	.0093292954	.68
.33	.0109266459	−.0957108458	.7714874242	.3799863433	−.0762248054	.0095352378	.67
.34	.0110646105	−.0966089124	.7615055448	.3922907352	−.0779855076	.0097335295	.66
.35	.0111908672	−.0974019922	.7513867969	.4045928906	−.0796925391	.0099239766	.65
.36	.0113054515	−.0980914176	.7411351552	.4168885248	−.0813441024	.0101063885	.64
.37	.0114084054	−.0986785435	.7307546195	.4291733480	−.0829384077	.0102805783	.63
.38	.0114997774	−.0991647468	.7202492136	.4414430664	−.0844736732	.0104463626	.62
.39	.0115796219	−.0995514258	.7096229842	.4536933833	−.0859481254	.0106035618	.61
.40	.0116480000	−.0998400000	.6988800000	.4659200000	−.0873600000	.0107520000	.60
.41	.0117049786	−.1000319092	.6880243508	.4781186167	−.0887075421	.0108915052	.59
.42	.0117506306	−.1001286132	.6770601464	.4902849336	−.0899890068	.0110219094	.58
.43	.0117850351	−.1001315915	.6659915155	.5024146520	−.0912026598	.0111430487	.57
.44	.0118082765	−.1000423424	.6548226048	.5145034752	−.0923467776	.0112547635	.56
.45	.0118204453	−.0998623828	.6435575781	.5265471094	−.0934196484	.0113568984	.55
.46	.0118216375	−.0995932476	.6322006152	.5385412648	−.0944195724	.0114493025	.54
.47	.0118119546	−.0992364892	.6207559108	.5504816567	−.0953448621	.0115318292	.53
.48	.0117915034	−.0987936768	.6092276736	.5623640064	−.0961938432	.0116043366	.52
.49	.0117603961	−.0982663965	.5976201254	.5741840421	−.0969648546	.0116666877	.51
.50	.0117187500	−.0976562500	.5859375000	.5859375000	−.0976525000	.0117187500	.50
	A_3	A_2	A_1	A_0	A_{-1}	A_{-2}	u

Note. If the arguments chosen are $x_1<x_2<x_3<x_4<x_5<x_6$ and interpolation is required at x such that $x_3<x<x_4$, compute $u = (x-x_3)/h$ where h is the interval of the argument. Then read the entries A_{-2}, A_{-1}, A_0, A_1, A_2 and A_3 corresponding to u and use the formula

$$A_{-2}\,f(x_1)+A_{-1}\,f(x_2)+A_0\,f(x_3)+A_1\,f(x_4)+A_2\,f(x_5)+A_3\,f(x_6).$$

For values of u in the right hand side column of the table, the coefficients are to be read as indicated in the bottom row of the table. Thus for $u = .59$, $A_{-2} = .0108915052$, $A_{-1} = -.0887075421$, $A_0 = .4781186167$, $A_1 = .6880243508$, $A_2 = -.1000319092$, $A_3 = .0117049786$,

15.1. Coefficients for Equispaced Ordinates

a. Introduction

For evaluating an integral $\int_c^d f(x)dx$ knowing only the values (ordinates) of $f(x)$ at equidistant values of x tabulated at intervals of h, the formula used is a weighted linear combination of the ordinates. Some well known and simple formulae are already given in Chapter **VI** of Part **I**. For a general formula using a polynomial approximation of the maximum degree for $f(x)$, the compounding coeffiients, which (apart from the multiplier h) depend upon the number and the position of the ordinates, are given in Table 15.1. As regards the position of ordinates, relative to interval (c, d), three types of situations are considered.

A. $(2m-1)$ internal and the two terminal ordinates at c and d.

B. $(2m-1)$ internal, two terminal and two external ordinates at $c-h$ and $d+h$.

C. $(2m-1)$ internal, two terminal and four external ordinates at $c-2h$, $c-h$, $d+h$ and $d+2h$.

Coefficients are given for $m = 1, 2, 3, 4$ and 5 in the case of A and B type of formulae and for $m = 1, 2, 3$, and 4 in the case of C type.

In Table 15.1, $f(a)$ is the ordinate at the midpoint a of the interval (c, d), $f(a\pm h)$ are the ordinates at the points $a+h$ and $a-h$ etc.

b. Application

To evaluate $\int_{2.5}^{4.5} \frac{1}{\sqrt{(1.5)}} e^{-x}\sqrt{x}\, dx$ using ordinates tabulated at an interval of 0.5.

Here $h = 0.5$ and and the number of internal and terminal ordinates available is 5 so that $2m+1 = 5$ or $m = 2$. The computations are as follows :

x	$f(x)$	coefficients from Table 15.1 for $m = 2$ for type of formula		
		A no external ordinate	B two external ordinates	C four external ordinates
1.5	0.308360	—	—	13
2.0	0.215963	—	−8	−224
$c = 2.5$	0.146450	14	342	5494
3.0	0.097304	64	1224	17632
$a = 3.5$	0.063746	24	664	10870
4.0	0.041335	64	1224	17632
$d = 4.5$	0.026591	14	342	5494
5.0	0.017001	—	−8	−224
5.5	0.010815	—	—	13
	divisor :	45	945	14175

Using A type formula the required integral is given by

$$h[14\times 0.146450+64\times 0.097304+\ldots] \div 45$$
$$= 0.5\times 12.825374 \div 45 = 0.142504.$$

TABLE 15.1. NUMERICAL INTEGRATION COEFFICIENTS

(Three-point to thirteen-point formulae with provision for using external ordinates)

m	integral	extra ordinates used	coefficient of $f(a)$	$f(a\pm h)$	$f(a\pm 2h)$	$f(a\pm 3h)$	$f(a\pm 4h)$	$f(a\pm 5h)$	$(a\pm 6h)$	divisor
1 (3)*	$\int_{a-h}^{a+h} f(x)dx$	A. no external	4	1						3
		B. 2 external	114	34	−1					90
		C. 4 external	4688	1503	−72	5				3780
2 (5)	$\int_{a-2h}^{a+2h} f(x)dx$	A. no external	24	64	14					45
		B. 2 external	664	1224	342	−8				945
		C. 4 external	10870	17632	5494	−224	13			14175
3 (7)	$\int_{a-3h}^{a+3h} f(x)dx$	A. no external	272	27	216	41				140
		B. 2 external	2090	774	1908	482	−9			1400
		C. 4 external	41192	21018	39696	11459	−388	19		30800
4 (9)	$\int_{a-4h}^{a+4h} f(x)dx$	A. no external	−18160	41984	−3712	23552	3956			14175
		B. 2 external	−2544	888192	161664	670656	154228	−2368		467775
		C. 4 external	250827024	994411008	356903280	854897920	228685476	−6534912	275216	638512875
5 (11)	$\int_{a-5h}^{a+5h} f(x)dx$	A. no external	2136840	−1302750	1362000	−242625	531500	80335		299376
		B. 2 external	3577456680	−2488912200	2153747250	−561143500	669257100	71569620	1346350	326918592

Note: To evaluate $\int_{a-mh}^{a+mh} f(x)dx$ by numerical integration choose the type of formula (A, B, C), compute the weighted sum of ordinates (values of $f(x)$) using the coefficients (weights) given in Table 15.1, multiply by h, the length of the interval of tabulation, and divide by the divisor in the last column. Note that $f(a)$ is the middle ordinate and that $f(a+ih)$ and $f(a-ih)$ have the same *weight coefficients*.

*The figure within brackets indicates the number of internal and terminal ordinates.

If ordinates at 2.0 and 5.0 are used in addition to internal and terminal ordinates (B type formula) the integral is

$$h[(-8)\times 0.215963+342\times 0.146450+\ldots] \div 945$$
$$=0.5\times 269.339118 \div 945 = 0.142507.$$

If ordinates at 1.5, 2.0, 5.0 and 5.5 are used in addition to internal and terminal ordinates, (C type formula) the integral is

$$h[13\times 0.308360+(-224)\times 0.215963+\ldots] \div 14175$$
$$= 0.5\times 4040.054461 \div 14175 \doteq 0.142506.$$

15.2. Abscissae and Weight Coefficients in Gaussian Quadrature Formulae

a. Introduction

The quadrature formulae given in Table 15.1 are useful when the values of the function to be integrated are known (tabulated) at equispaced values of the abscissa. But if such a table is not available and the function itself has to be evaluated at selected values of the abscissa, one can use more precise quadrature formulae due to Gauss, which specify an optimum choice of the abscissa for this purpose. To apply the formulae given in Tables 15.2.—15.4, the values of the function are computed at the specified values of the abscissa and then a linear combination of these values is taken using the weight coefficients.

b. n-point Gauss-Legendre formula

$$\int_{-1}^{1} f(x)dx = g_1 f(x_1)+g_2 f(x_2)+\ldots+g_n f(x_n).$$

This formula is useful for evaluating definite integrals of the type $\int_{-1}^{1} f(x)dx$. The values of x where the function $f(x)$ has to be evaluated and the corresponding coefficients g are given in Table 15.2, for any chosen value of $n = 2(1)16$. Note that integration in any finite range can be reduced to integration over the range $(-1, 1)$ by suitable transformation of the variable, so that Table 15.2 is useful in evaluating integrals of the form $\int_a^b f(x)dx$.

c. n-point Gauss-Laguerre formula

$$\int_0^{\infty} e^{-x} f(x)dx = l_1 f(x_1)+l_2 f(x_2)+\ldots+l_n f(x_n).$$

This formula is useful for evaluating definite integrals of the type $\int_0^{\infty} e^{-x}f(x)dx$. The values of x where the function $f(x)$ has to be evaluated together with the corresponding coefficients l are given in Table 15.3, for any chosen value of $n = 2(1)10$.

d. n-point Gauss-Hermite formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = h_1 f(x_1)+h_2 f(x)+\ldots+h_n f(x_n).$$

This formula is useful for evaluating definite integrals of the type $\int_{-\infty}^{\infty} e^{-x^2}f(x)dx$. The values of x where the function $f(x)$ has to be evaluated together with the corresponding coefficients h are given in Table 15.4, for any chosen value of $n = 2(1)10$.

TABLE 15.2. GAUSS-LEGENDRE QUADRATURE FORMULA: ABSCISSAE AND WEIGHT COEFFICIENTS

[Note that the abscissae chosen are symmetrical about the origin. Abscissae with the same magnitude but of opposite sign have the same weight coefficients].

$\pm x$	g
$n = 2$	
0.57735 02692	1.00000 00000
$n = 5$	
0.00000 00000	0.56888 88889
0.53846 93101	0.47862 86705
0.90617 98459	0.23692 68851
$n = 8$	
0.18343 46425	0.36268 37834
0.52553 24099	0.31370 66459
0.79666 64774	0.22238 10345
0.96028 98565	0.10122 85363
$n = 11$	
0.00000 00000	0.27292 50868
0.26954 31560	0.26280 45445
0.51909 61291	0.23319 37646
0.73015 20056	0.18629 02109
0.88706 25998	0.12558 03695
0.97822 86581	0.05566 85671
$n = 14$	
0.10805 49487	0.21526 38535
0.31911 23689	0.20519 84637
0.51524 86364	0.18553 83975
0.68729 29048	0.15720 31672
0.82720 13151	0.12151 85707
0.92843 48837	0.08015 80872
0.98628 38087	0.03511 94603

$\pm x$	g
$n = 3$	
0.00000 00000	0.88888 88889
0.77459 66692	0.55555 55556
$n = 6$	
0.23861 91861	0.46791 39346
0.66120 93865	0.36076 15730
0.93246 95142	0.17132 44924
$n = 9$	
0.00000 00000	0.33023 93550
0.32425 34234	0.31234 70770
0.61337 14327	0.26061 06964
0.83603 11073	0.18064 81607
0.96816 02395	0.08127 43884
$n = 12$	
0.12533 34085	0.24914 70458
0.36783 14989	0.23349 25365
0.58731 79543	0.20316 74267
0.76990 26742	0.16007 83285
0.90411 72564	0.10693 93260
0.98156 06342	0.04717 53364
$n = 15$	
0.00000 00000	0.20257 82419
0.20119 40940	0.19843 14853
0.39415 13471	0.18616 10000
0.57097 21726	0.16626 92058
0.72441 77314	0.13957 06779
0.84820 65834	0.10715 92205
0.93727 33924	0.07036 60475
0.98799 25180	0.03075 32420

$\pm x$	g
$n = 4$	
0.33998 10436	0.65214 51549
0.86113 63116	0.34785 48451
$n = 7$	
0.00000 00000	0.41795 91837
0.40584 51514	0.38183 00505
0.74153 11856	0.27970 53915
0.94910 79123	0.12948 49662
$n = 10$	
0.14887 43390	0.29552 42247
0.43339 53941	0.26926 02109
0.67940 95683	0.21908 63625
0.86506 33667	0.14945 13492
0.97390 65285	0.06667 13443
$n = 13$	
0.00000 00000	0.23255 15532
0.23045 83160	0.22628 31803
0.44849 27510	0.20781 60475
0.64234 93394	0.17814 59808
0.80157 80907	0.13887 35102
0.91759 83992	0.09212 14998
0.98418 30547	0.04048 40048
$n = 16$	
0.09501 25098	0.18945 06105
0.28160 35508	0.18260 34150
0.45801 67777	0.16915 65194
0.61787 62444	0.14959 59888
0.75540 44084	0.12462 89713
0.86563 12024	0.09515 85117
0.94457 50231	0.06225 35239
0.98940 09350	0.02715 24594

TABLE 15.3. GAUSS-LAGUERRE QUADRATURE FORMULA : ABSCISSAE AND WEIGHT COEFFICIENTS

$n = 2$

x	l
0.58578 64376	0.85355 33906
3.41421 35624	0.14644 66094

$n = 3$

x	l
0.41577 45568	0.71109 30099
2.29428 03603	0.27851 77336
6.28994 50829	0.01038 92565

$n = 4$

x	l
0.32254 76896	0.60315 41043
1.74576 11012	0.35741 86924
4.53662 02969	0.03888 79085
9.39507 09123	$0.0^{3}539$ 29471

$n = 5$

x	l
0.26356 03197	0.52175 56106
1.41340 30591	0.39866 68111
3.59642 57710	0.07594 24497
7.08581 00059	$0.0^{2}361$ 17587
12.64080 08443	$0.0^{4}233$ 69972

$n = 6$

x	l
0.22284 66042	0.45896 46740
1.18893 21017	0.41700 08308
2.99273 63261	0.11337 33821
5.77514 35691	0.01039 91975
9.83746 74184	$0.0^{3}261$ 01720
15.98287 39806	$0.0^{6}898$ 54791

$n = 7$

x	l
0.19304 36766	0.40931 89517
1.02666 48953	0.42183 12779
2.56787 67450	0.14712 63487
4.90035 30845	0.02063 35145
8.18215 34446	$0.0^{2}107$ 40101
12.73418 02918	$0.0^{4}158$ 65464
19.39572 78623	$0.0^{7}317$ 03155

$n = 8$

x	l
0.17027 96323	0.36918 85893
0.90370 17768	0.41878 67808
2.25108 66299	0.17579 49866
4.26670 01703	0.03334 34923
7.04590 54024	$0.0^{2}279$ 45362
10.75851 60102	$0.0^{4}907$ 65088
15.74067 86413	$0.0^{6}848$ 57467
22.86313 17369	$0.0^{8}104$ 80012

$n = 9$

x	l
0.15232 22277	0.33612 64218
0.80722 00227	0.41121 39804
2.00513 51556	0.19928 75254
3.78347 39733	0.04746 05628
6.20495 67779	$0.0^{2}559$ 96266
9.37298 52517	$0.0^{3}305$ 24977
13.46623 69111	$0.0^{5}659$ 21230
18.83359 77890	$0.0^{7}411$ 07693
26.37407 18909	$0.0^{10}32$ 90874

$n = 10$

x	l
0.13779 34705	0.30844 11158
0.72945 45495	0.40111 99292
1.80834 29027	0.21806 82876
3.40143 36979	0.06208 74561
5.55249 61400	$0.0^{2}950$ 15170
8.33015 27468	$0.0^{3}753$ 00839
11.84378 58379	$0.0^{4}282$ 59233
16.27925 78314	$0.0^{6}424$ 93140
21.99658 58120	$0.0^{8}183$ 95648
29.92069 70123	$0.0^{12}99$ 11827

TABLE 15.4. GAUSS—HERMITE QUADRATURE FORMULA: ABSCISSAE AND WEIGHT COEFFICIENTS

[Note that the abscissae chosen are symmetrical about the origin. Abscissae with the same magnitude but of opposite sign have the same weight coefficients].

$\pm x$	h
$n = 2$	
0.70710 67812	0.88622 69255
$n = 3$	
0.00000 00000	1.18163 59006
1.22474 48714	0.29540 89752
$n = 4$	
0.52464 76233	0.80491 40900
1.65068 01239	0.08131 28354
$n = 5$	
0.00000 00000	0.94530 87205
0.95857 24646	0.39361 93232
2.02018 28705	0.01995 32421
$n = 6$	
0.43607 74119	0.72462 95952
1.33584 90740	0.15706 73203
2.35060 49737	$0.0^{2}453$ 00099
$n = 7$	
0.00000 00000	0.81026 46176
0.81628 78829	0.42560 72526
1.67355 16288	0.05451 55828
2.65196 13568	$0.0^{3}971$ 78125
$n = 8$	
0.38118 69902	0.66114 70126
1.15719 37124	0.20780 23258
1.98165 67567	0.01707 79830
2.93063 74203	$0.0^{3}199$ 60407
$n = 9$	
0.00000 00000	0.72023 52156
0.72355 10188	0.43265 15590
1.46855 32892	0.08847 45274
2.26658 05845	$0.0^{2}494$ 36243
3.19099 32018	$0.0^{4}396$ 06977
$n = 10$	
0.34290 13272	0.61086 26337
1.03661 08298	0.24013 86111
1.75668 36493	0.03387 43945
2.53273 16742	$0.0^{2}134$ 36457
3.43615 91188	$0.0^{5}764$ 04329
$n = 11$	
0.00000 00000	0.65475 92869
0.65680 95669	0.42935 97524
1.32655 70845	0.11722 78752
2.02594 80158	0.01191 13954
2.78329 00998	$0.0^{3}346$ 81947
3.66847 08466	$0.0^{5}143$ 95604
$n = 12$	
0.31424 03763	0.57013 52363
0.94778 83912	0.26049 23103
1.59768 26352	0.05160 79856
2.27950 70805	$0.0^{2}390$ 53906
3.02063 70251	$0.0^{4}857$ 36870
3.88972 48979	$0.0^{6}265$ 85517
$n = 13$	
0.00000 00000	0.60439 31879
0.60576 38792	0.42161 62969
1.22005 50366	0.14032 33207
1.85310 76516	0.02086 27753
2.51973 56857	$0.0^{2}120$ 74600
3.24660 89784	$0.0^{4}204$ 30360
4.10133 75962	$0.0^{7}482$ 57319
$n = 14$	
0.29174 55107	0.53640 59097
0.87871 37873	0.27310 56091
1.47668 27311	0.06850 55342
2.09518 32585	$0.0^{2}785$ 00547
2.74847 07250	$0.0^{3}355$ 09261
3.46265 69336	$0.0^{5}471$ 64844
4.30444 85705	$0.0^{8}862$ 85912
$n = 15$	
0.00000 00000	0.56410 03087
0.56506 95833	0.41202 86875
1.13611 55852	0.15848 89158
1.71999 25752	0.03078 00339
2.32573 24862	$0.0^{2}277$ 80688
2.96716 69279	$0.0^{3}100$ 00444
3.66995 03734	$0.0^{5}105$ 91155
4.49999 07073	$0.0^{8}152$ 24758
$n = 16$	
0.27348 10461	0.50792 94790
0.82295 14491	0.28064 74585
1.38025 85392	0.08381 00414
1.95178 79909	0.01288 03115
2.54620 21578	$0.0^{3}932$ 28401
3.17699 91620	$0.0^{4}271$ 18601
3.86944 79049	$0.0^{6}232$ 09808
4.68873 89393	$0.0^{9}265$ 48075

a. Introduction

Consider the following polynomials due to Tchebycheff

$$\phi_0(x) = 1$$

$$\phi_1(x) = x$$

$$\phi_2(x) = x^2 - \frac{(n^2-1)}{12}$$

$$\phi_3(x) = x^3 - \frac{(3n^2-7)x}{20}$$

$$\phi_4(x) = x^4 - \frac{(3n^2-13)x^2}{14} + \frac{3(n^2-1)(n^2-9)}{560}$$

$$\phi_5(x) = x^5 - \frac{5(n^2-7)x^3}{18} + \frac{(15n^4-230n^2+407)x}{1008}$$

$$\phi_6(x) = x^6 - \frac{5(3n^2-31)x^4}{44} + \frac{(5n^4-110n^2+329)x^2}{176}$$

$$- \frac{5(n^2-1)(n^2-9)(n^2-25)}{14784}.$$

Observe that $$\phi_j(x) = (-1)^j \phi_j(-x).$$

These polynomials have the following orthogonality property

If x ranges over the n values $t - \frac{n+1}{2}$ $(t = 1, 2, \ldots, n)$, n being an integer

$$\sum_x \phi_i(x)\,\phi_j(x) = 0 \quad \text{whenever } i \neq j.$$

Let $\sum_x [\phi_i(x)]^2 = A_i$. Then the first six values of A_i are

$$A_0 = n$$

$$A_1 = n(n^2-1)/12$$

$$A_2 = n(n^2-1)(n^2-4)/180$$

$$A_3 = n(n^2-1)(n^2-4)(n^2-9)/2800$$

$$A_4 = n(n^2-1)(n^2-4)(n^2-9)(n^2-16)/44100$$

$$A_5 = n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)/698544$$

$$A_6 = n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)(n^2-36)/11099088.$$

Values of $\phi_i(x)$ with some modification (see iii below) are given in Table 16.1 for $i = 1(1)5$ and $n = 3(1)30$. *The following points should be noted.*

(i) The table provides polynomial values only for those n values of x given by $x = t - \frac{n+1}{2}$, $(t = 1, 2, \ldots, n)$.

(ii) To save space, however, for values of $n \geqslant 13$, arguments covering the half range corresponding to $t = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$ only are given; values for the other half are to be obtained from the symmetry (antisymmetry) relation, $\phi_j(x) = (-1)^j \phi_j(-x)$.

(iii) To avoid fractional values, the polynomials $\xi_i(x) = \lambda_i \phi_i(x)$ instead of $\phi_i(x)$ have been tabulated and the constants λ_i are shown in the bottom line of each section of Table 16.1. The line just above the bottom line shows values of $\lambda_i^2 A_i = B_i$. Thus to obtain the value of $\phi_i(x)$, if necessary, the tabulated value $\xi_i(x)$ has to be divided by λ_i. Such a computation is unnecessary in practice, and one can use the values of $\xi_i(x)$ directly as shown in the illustrative example.

(iv) The argument x is not explicitly shown in the table but the ξ_1 column, in fact, gives x for odd values of n and $2x$ for even values of n.

The tabulated values are useful in fitting polynomials of successive degrees, in stages, if necessary, to observed data. The values of x, the abscissa at which the argument y is observed, should, however, be at equal intervals.

b. Application

An experiment was conducted in a randomised block layout to test whether subjecting seeds to a temperature treatment before planting has any effect on yield. Data on yield per plot at various levels of temperature for seed treatment are summarised as follows:

temperature (°F)	60	75	90	105	120
mean yield	60.74	80.00	87.90	89.48	80.60

ANALYSIS OF VARIANCE

(for randomised block design)

source	d.f.	s.s.	m.s.	F
blocks	4	877.56	219.39	86.37
treatments	4	2616.30	654.07	257.51
error	16	40.60	2.54	

Analyse the results to find the optimum temperature for treatment of seeds.

(i) *Fitting a polynomial regression (upto fourth degree) of mean yield per plot on temperature*

All the successive four stages of fitting the polynomial giving the regression coefficients on the linear, quadratic, cubic and quartic terms are shown below :

temperature	mean yield y	from Table 16.1 for $n = 5$ ξ_1	ξ_2	ξ_3	ξ_4
60	60.74	−2	2	−1	1
75	80.00	−1	−1	2	−4
90	87.90	0	−2	0	6
105	89.48	1	−1	−2	−4
120	80.60	2	2	1	1
	$\Sigma y\xi$	49.20	−62.60	0.90	−9.18
	B	10	14	10	70
regression coefficient $\Sigma y\xi/B$		4.92	−4.4714	0.09	−0.1311
sum of squares due to regression $(\Sigma y\xi)^2/B$		242.064	279.912	0.081	1.204

Thus we have the ANOVA table for testing the significance of the regression coefficients.

Since y is the mean of 5 observations each sum of squares given in the last row of the above table is multiplied by 5 for purpose of analysis of variance test.

source	d.f.	s.s.	m.s.	m.s.	s.s.	d.f.	source
linear	1	1210.32	1210.32	468.66	1405.98	3	residual 1
quadratic	1	1399.56	1399.56	3.21	6.42	2	residual 2
cubic	1	0.40	0.40	6.02	6.02	1	residual 3
quartic	1	6.02	6.02				
total (treatments)	4	2616.30					
error	16	40.60	2.54	2.54	40.60	16	error

The residual after fitting the linear terms is $2616.30-1210.32 = 1405.98$ on 3 d.f. Similarly the residual after fitting the linear and quadratic terms is $1405.98-1399.56 = 6.42$ and so on. Each residual is tested against error, successively starting from residual 1. Residual 2 is unimportant since the variance ratio $3.21/2.54$ is not large enough on 2 and 16 d.f. We may normally stop at this stage and infer that a quadratic fit is sufficient.

The equation to the parabola is (using the regression coefficients computed earlier),

$$Y = 79.744+4.92\xi_1-4.4714\xi_2.$$

Since $\xi_1 = x = (t-90)/15,$

$$\xi_2 = x^2-2 = [(t-90)^2/225]-2,$$

we have $Y = 88.6868+0.3280(t-90)-0.0199\,(t-90)^2.$

By equating the derivative with respect to t to zero

$$0.0398(t-90) = 0.3280$$

or, the maximum of Y is attained at $t = 90+8.24 = 98.24^\circ F.$

(ii) *Standard error of an estimated yield*

The estimated mean yield at temperature $t = 80°F$ (say) is given by

$$79.744+4.92\xi_1-4.4714\xi_2 = 83.42$$

where
$$\xi_1 = \frac{80-90}{15} = -0.6667$$

and
$$\xi_2 = \xi_1^2-2 = -1.5556.$$

The sampling variance of the estimate is

$$\sigma^2\left[\frac{1}{n}+\frac{\xi_1^2}{B_1}+\frac{\xi_2^2}{B_2}\right] = \sigma^2(0.0400+0.0444+0.1728)$$
$$= 0.2572\sigma^2.$$

(It may be noted that the variance of an individual regression coefficient b_i is σ^2/B_i and that the b's are mutually uncorrelated).

(iii) *Confidence interval for temperature τ at which yield is a maximum*

The value of τ is given by the equation

$$\frac{b_1}{15}+\frac{2b_2}{225}(\tau-90) = 0.$$

The sampling variance of the expression on the left hand side is

$$\sigma^2\left[\frac{1}{(15)^2B_1}+\frac{4(\tau-90)^2}{(225)^2B_2}\right]$$

Consider the inequality

$$\frac{\left|\frac{b_1}{15}+\frac{2b_2}{225}(\tau-90)\right|}{\sqrt{\frac{1}{B_1(15)^2}+\frac{4(\tau-90)^2}{B_2(225)^2}}} \leqslant 2.120s$$

where s^2 is the estimate of σ^2 (the error *m.s.* in the ANOVA table, with 16 d.f.) and 2.120 is the 5% point of Student's t with 16 d.f. This leads to a quadratic in $(\tau-90)$, whose roots provide 95% confidence limits for τ. In this particular example the limits are 96.08 and 101.13.

c. Some other tables

1. Fisher, R. A. and Yates, F. (1957): *Statistical Tables for Biological, Agricultural and Medical Research.* (5th edition), Oliver and Boyd, London. (Table XXIII),
[$n = 3(1)\,45$, $r = 1(1)\,5$
$n = 46(1)\,75$, $r = 2(1)5$].

2. Pearson, E. S. and Hartley, H. O. (1957): *Biometrika Tables for Statisticians*, Biometrika Trust, Cambridge University Press. (Table 47),
[$n = 3(1)\,52$, $r = 1(1)\,6$].

3. Anderson, R. L. and Houseman, E. E. (1942): *Tables of Orthogonal Polynomial Values Extended to $n = 104$.* Iowa State College, Agricultural Experiment Station, Bulletin 297.
[$n = 3(1)\,104$, $r = 1(1)\,5$].

4. DeLury, D. B. (1950): *Values and Integrals of the Orthogonal Polynomials up to $n = 26$.* University of Toronto Press.
[$n = 3(1)\,26$. $r = 1(1)\,25$].

TABLE 16.1. ORTHOGONAL POLYNOMIALS

From $n = 13$, the polynomial values are tabulated for the first $\left[\frac{n+1}{2}\right]$ values of the argument. The other values are obtained by symmetry for the even order polynomials and antisymmetry for the odd order polynomials. Note that $\xi_i(x) = (-1)^i \xi_i(-x)$.

$n = 3$

	ξ_1	ξ_2
	−1	1
	0	−2
	1	1
B:	2	6
λ:	1	3

$n = 4$

	ξ_1	ξ_2	ξ_3
	−3	1	−1
	−1	−1	3
	1	−1	−3
	3	1	1
B:	20	4	20
λ:	2	1	$\frac{10}{3}$

$n = 5$

	ξ_1	ξ_2	ξ_3	ξ_4
	−2	2	−1	1
	−1	−1	2	−4
	0	−2	0	6
	1	−1	−2	−4
	2	2	1	1
B:	10	14	10	70
λ:	1	1	$\frac{5}{6}$	$\frac{35}{12}$

$n = 6$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−5	5	−5	1	−1
	−3	−1	7	−3	5
	−1	−4	4	2	−10
	1	−4	−4	2	10
	3	−1	−7	−3	−5
	5	5	5	1	1
B:	70	84	180	28	252
λ:	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{21}{10}$

$n = 7$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−3	5	−1	3	−1
	−2	0	1	−7	4
	−1	−3	1	1	−5
	0	−4	0	6	0
	1	−3	−1	1	5
	2	0	−1	−7	−4
	3	5	1	3	1
B:	28	84	6	154	84
λ:	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{20}$

$n = 8$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−7	7	−7	7	−7
	−5	1	5	−13	23
	−3	−3	7	−3	−17
	−1	−5	3	9	−15
	1	−5	−3	9	15
	3	−3	−7	−3	17
	5	1	−5	−13	−23
	7	7	7	7	7
B:	168	168	264	616	2184
λ:	2	1	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{7}{10}$

$n = 9$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−4	28	−14	14	−4
	−3	7	7	−21	11
	−2	−8	13	−11	−4
	−1	−17	9	9	−9
	0	−20	0	18	0
	1	−17	−9	9	9
	2	−8	−13	−11	4
	3	7	−7	−21	−11
	4	28	14	14	4
B:	60	2772	990	2002	468
λ:	1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{20}$

$n = 10$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−9	6	−42	18	−6
	−7	2	14	−22	14
	−5	−1	35	−17	−1
	−3	−3	31	3	−11
	−1	−4	12	18	−6
	1	−4	−12	18	6
	3	−3	−31	3	11
	5	−1	−35	−17	1
	7	2	−14	−22	−14
	9	6	42	18	6
B:	330	132	8580	2860	780
λ:	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{5}{12}$	$\frac{1}{10}$

$n = 11$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−5	15	−30	6	−3
	−4	6	6	−6	6
	−3	−1	22	−6	1
	−2	−6	23	−1	−4
	−1	−9	14	4	−4
	0	−10	0	6	0
	1	−9	−14	4	4
	2	−6	−23	−1	4
	3	−1	−22	−6	−1
	4	6	−6	−6	−6
	5	15	30	6	3
B:	110	858	4290	286	156
λ:	1	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{40}$

$n = 12$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−11	55	−33	33	−33
	−9	25	3	−27	57
	−7	1	21	−33	21
	−5	−17	25	−13	−29
	−3	−29	19	12	−44
	−1	−35	7	28	−20
	1	−35	−7	28	20
	3	−29	−19	12	44
	5	−17	−25	−13	29
	7	1	−21	−33	−21
	9	25	−3	−27	−57
	11	55	33	33	33
B:	572	12012	5148	8008	15912
λ:	2	3	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{3}{20}$

B = sum of squares of the n values of the polynomial

λ = divisor for the coefficients ($\phi(x) = \xi(x)/\lambda$)

TABLE 16.1. (*continued*). ORTHOGONAL POLYNOMIALS

$n = 13$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−6	22	−11	99	−22
	−5	11	0	−66	33
	−4	2	6	−96	18
	−3	−5	8	−54	−11
	−2	−10	7	11	−26
	−1	−13	4	64	−20
	0	−14	0	84	0
B:	182	2002	572	68068	6188
λ:	1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{120}$

$n = 14$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−13	13	−143	143	−143
	−11	7	−11	−77	187
	−9	2	66	−132	132
	−7	−2	98	−92	−28
	−5	−5	95	−13	−139
	−3	−7	67	63	−145
	−1	−8	24	108	−60
B:	910	728	97240	136136	235144
λ:	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{7}{30}$

$n = 15$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−7	91	−91	1001	−1001
	−6	52	−13	−429	1144
	−5	19	35	−869	979
	−4	−8	58	−704	44
	−3	−29	61	−249	−751
	−2	−44	49	251	−1000
	−1	−53	27	621	−675
	0	−56	0	756	0
B:	280	37128	39780	6466460	10581480
λ:	1	3	$\frac{5}{6}$	$\frac{35}{12}$	$\frac{21}{20}$

$n = 16$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−15	35	−455	273	−143
	−13	21	−91	−91	143
	−11	9	143	−221	143
	−9	−1	267	−201	33
	−7	−9	301	−101	−77
	−5	−15	265	23	−131
	−3	−19	179	129	−115
	−1	−21	63	189	−45
B:	1360	5712	1007760	470288	201552
λ:	2	1	$\frac{10}{3}$	$\frac{7}{12}$	$\frac{1}{10}$

$n = 17$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−8	40	−28	52	−104
	−7	25	−7	−13	91
	−6	12	7	−39	104
	−5	1	15	−39	39
	−4	−8	18	−24	−36
	−3	−15	17	−3	−83
	−2	−20	13	17	−88
	−1	−23	7	31	−55
	0	−24	0	36	0
B:	408	7752	3876	16796	100776
λ:	1	1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$

$n = 18$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−17	68	−68	68	−884
	−15	44	−20	−12	676
	−13	23	13	−47	871
	−11	5	33	−51	429
	−9	−10	42	−36	−156
	−7	−22	42	−12	−588
	−5	−31	35	13	−733
	−3	−87	23	33	−583
	−1	−40	8	44	−220
B:	1938	23256	23256	28424	6953544
λ:	2	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{3}{10}$

$n = 19$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−9	51	−204	612	−102
	−8	34	−68	−68	68
	−7	19	28	−388	98
	−6	6	89	−453	58
	−5	−5	120	−354	−3
	−4	−14	126	−168	−54
	−3	−21	112	42	−79
	−2	−26	83	227	−74
	−1	−29	44	352	−44
	0	−30	0	396	0
B:	570	13566	213180	2288132	89148
λ:	1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{1}{40}$

$n = 20$

	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
	−19	57	−969	1938	−1938
	−17	39	−357	−102	1122
	−15	23	85	−1122	1802
	−13	9	377	−1402	1222
	−11	−3	539	−1187	187
	−9	−13	591	−687	−771
	−7	−21	553	−77	−1351
	−5	−27	445	503	−1441
	−3	−31	287	948	−1076
	−1	−33	99	1188	−396
B:	2660	17556	4903140	22881320	31201800
λ:	2	1	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{7}{20}$

TABLE 16.1. (*continued*). ORTHOGONAL POLYNOMIALS

$n = 21$					$n = 22$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−10	190	−285	969	−3876	−21	35	−133	1197	−2261
−9	133	−114	0	1938	−19	25	−57	57	969
−8	82	12	−510	3468	−17	16	0	−570	1938
−7	37	98	−680	2618	−15	8	40	−810	1598
−6	−2	149	−615	788	−13	1	65	−775	663
−5	−35	170	−406	−1063	−11	−5	77	−563	−363
−4	−62	166	−130	−2354	−9	−10	78	−258	−1158
−3	−83	142	150	−2819	−7	−14	70	70	−1554
−2	−98	103	385	−2444	−5	−17	55	365	−1509
−1	−107	54	540	−1404	−3	−19	35	585	−1079
0	−110	0	594	0	−1	−20	12	702	−390
B : 770	201894	432630	5720330	121687020	3542	7084	96140	8748740	40562340
λ : 1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{21}{40}$	2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{7}{12}$	$\frac{7}{30}$

$n = 23$					$n = 24$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−11	77	−77	1463	−209	−23	253	−1771	253	−4807
−10	56	−35	133	76	−21	187	−847	33	1463
−9	37	−3	−627	171	−19	127	−133	−97	3743
−8	20	20	−950	152	−17	73	391	−157	3553
−7	5	35	−955	77	−15	25	745	−165	2071
−6	−8	43	−747	−12	−13	−17	949	−137	169
−5	−19	45	−417	−87	−11	−53	1023	−87	−1551
−4	−28	42	−42	−132	−9	−83	987	−27	−2721
−3	−35	35	315	−141	−7	−107	861	33	−3171
−2	−40	25	605	−116	−5	−125	665	85	−2893
−1	−43	13	793	−65	−3	−137	419	123	−2005
0	−44	0	858	0	−1	−143	143	143	−715
B : 1012	35420	32890	13123110	340860	4600	394680	17760600	394680	177928920
λ : 1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{60}$	2	3	$\frac{10}{3}$	$\frac{1}{12}$	$\frac{3}{10}$

$n = 25$					$n = 26$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−12	92	−506	1518	−1012	−25	50	−1150	2530	−2530
−11	69	−253	253	253	−23	38	−598	506	506
−10	48	−55	−517	748	−21	27	−161	−759	1771
−9	29	93	−897	753	−19	17	171	−1419	1881
−8	12	196	−982	488	−17	8	408	−1614	1326
−7	−3	259	−857	119	−15	0	560	−1470	482
−6	−16	287	−597	−236	−13	−7	637	−1099	−377
−5	−27	285	−267	−501	−11	−13	649	−599	−1067
−4	−36	258	78	−636	−9	−18	606	−54	−1482
−3	−43	211	393	−631	−7	−22	518	466	−1582
−2	−48	149	643	−500	−5	−25	395	905	−1381
−1	−51	77	803	−275	−3	−27	247	1221	−935
0	−52	0	858	0	−1	−28	84	1386	−330
B : 1300	53820	1480050	14307150	7803900	5850	16380	7803900	40060020	48384180
λ : 1	1	$\frac{5}{6}$	$\frac{5}{12}$	$\frac{1}{20}$	2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{1}{10}$

TABLE 16.1. (*continued*). ORTHOGONAL POLYNOMIALS

$n = 27$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−13	325	−130	2990	−16445
−12	250	−70	690	2530
−11	181	−22	−782	10879
−10	118	15	−1587	12144
−9	61	42	−1872	9174
−8	10	60	−1770	4188
−7	−35	70	−1400	−1162
−6	−74	73	−867	−5728
−5	−107	70	−262	−8803
−4	−134	62	338	−10058
−3	−155	50	870	−9479
−2	−170	35	1285	−7304
−1	−179	18	1548	−3960
0	−182	0	1638	0
B: 1638	712530	101790	56448210	2032135560
λ: 1	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{21}{40}$

$n = 28$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−27	117	−585	1755	−13455
−25	91	−325	455	1495
−23	67	−115	−395	8395
−21	45	49	−879	9821
−19	25	171	−1074	7866
−17	7	255	−1050	4182
−15	−9	305	−870	22
−13	−23	325	−590	−3718
−11	−35	319	−259	−6457
−9	−45	291	81	−7887
−7	−53	245	395	−7931
−5	−59	185	655	−6701
−3	−63	115	840	−4456
−1	−65	39	936	−1560
B: 7308	95004	2103660	19634160	1354757040
λ: 2	1	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{7}{20}$

$n = 29$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−14	126	−819	4095	−8190
−13	99	−468	1170	585
−12	74	−182	−780	4810
−11	51	44	−1930	5885
−10	30	215	−2441	4958
−9	11	336	−2460	2946
−8	−6	412	−2120	556
−7	−21	448	−1540	−1694
−6	−34	449	−825	−3454
−5	−45	420	−66	−4521
−4	−54	366	660	−4818
−3	−61	292	1290	−4373
−2	−66	203	1775	−3298
−1	−69	104	2080	−1768
0	−70	0	2184	0
B: 2030	113274	4207320	107987880	500671080
λ: 1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{7}{40}$

$n = 30$				
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
−29	203	−1827	23751	−16965
−27	161	−1071	7371	585
−25	122	−450	−3744	9360
−23	86	46	−10504	11960
−21	53	427	−13749	10535
−19	23	703	−14249	6821
−17	−4	884	−12704	2176
−15	−28	980	−9744	−2384
−13	−49	1001	−5929	−6149
−11	−67	957	−1749	−8679
−9	−82	858	2376	−9768
−7	−94	714	6096	−9408
−5	−103	535	9131	−7753
−3	−109	331	11271	−5083
−1	−112	112	12376	−1768
B: 8990	302064	21360240	3671587920	2145733200
λ: 2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{35}{12}$	$\frac{3}{10}$

Table 17.1 gives the squares of natural numbers upto 999. The same table can be used to find approximate square roots of numbers, correct upto 3 significant digits, by reading in the reverse way. If x is the given number and x_0 is the approximate square root read from Table 17 1, then a second approximation correct upto 6 significant digits is

$$x_1 = \frac{1}{2}\left(x_0 + \frac{x}{x_0}\right)$$

and a third approximation correct to 12 significant digits is

$$x_2 = \frac{1}{2}\left(x_1 + \frac{x}{x_1}\right)$$

Example 1. To compute $\sqrt{83}$.

To make an effective use of the Table we find $\sqrt{830000}$, making a 6 digited number, and divide the result by 100. From Table 17.1 we find that $911^2 = 829921$ closest to 830000, so that 911 is a first approximation. The second approximation is

$$\frac{1}{2}\left(911 + \frac{830000}{911}\right) = 911.043$$

Dividing by 100, $\sqrt{83} = 9.11043$ correct to six significant digits.

Example 2. To compute $\sqrt{831}$.

Since the number of digits is odd, we consider the five digited number 83100 multiplying the original number by hundred. Now $288^2 = 82944$, so that $x_0 = 288$ and

$$x_1 = \frac{1}{2}\left(288 + \frac{83100}{288}\right) = 288.144$$

Dividing by 10, $\sqrt{831} = 28.8144$ correct to six significant figures.

Example 3. To compute $\sqrt{7134268.17}$.

Since $267^2 = 71289$, we take $x_0 = 2670$. The second approximation is

$$\frac{1}{2}\left(2670 + \frac{7134268.17}{2670}\right) = 2671.01 \text{ (correct to six digits).}$$

Example 4. To compute $\sqrt{71342681.7}$.

Since $845^2 = 714025$, we take $x_0 = 8450$. The second approximation is

$$\frac{1}{2}\left(8450 + \frac{71342681.7}{8450}\right) = 8446.46 \text{ (correct to six digits).}$$

Table 17.3 is similarly useful in finding cube roots. Thus if it be required to find the cube root of a number x, we find from Table 17.3 the two digited number x_0 whose cube is closest to x. The second approximation is $x_1 = \frac{1}{3}\left(2x_0 + \frac{n}{x_0^2}\right)$ correct to four significant digits.

Some tables in this Chapter are not preceded by notes. Such tables are self-explanatory.

TABLE 17.1. SQUARES OF NATURAL NUMBERS

n	n^2	n	n^2	n	n^2	n	n^2	n	n^2
1	1	51	2601	101	10201	151	22801	201	40401
2	4	52	2704	102	10404	152	23104	202	40804
3	9	53	2809	103	10609	153	23409	203	41209
4	16	54	2916	104	10816	154	23716	204	41616
5	25	55	3025	105	11025	155	24025	205	42025
6	36	56	3136	106	11236	156	24336	206	42436
7	49	57	3249	107	11449	157	24649	207	42849
8	64	58	3364	108	11664	158	24964	208	43264
9	81	59	3481	109	11881	159	25281	209	43681
10	100	60	3600	110	12100	160	25600	210	44100
11	121	61	3721	111	12321	161	25921	211	44521
12	144	62	3844	112	12544	162	26244	212	44944
13	169	63	3969	113	12769	163	26569	213	45369
14	196	64	4096	114	12996	164	26896	214	45796
15	225	65	4225	115	13225	165	27225	215	46225
16	256	66	4356	116	13456	166	27556	216	46656
17	289	67	4489	117	13689	167	27889	217	47089
18	324	68	4624	118	13924	168	28224	218	47524
19	361	69	4761	119	14161	169	28561	219	47961
20	400	70	4900	120	14400	170	28900	220	48400
21	441	71	5041	121	14641	171	29241	221	48841
22	484	72	5184	122	14884	172	29584	222	49284
23	529	73	5329	123	15129	173	29929	223	49729
24	576	74	5476	124	15376	174	30276	224	50176
25	625	75	5625	125	15625	175	30625	225	50625
26	676	76	5776	126	15876	176	30976	226	51076
27	729	77	5929	127	16129	177	31329	227	51529
28	784	78	6084	128	16384	178	31684	228	51984
29	841	79	6241	129	16641	179	32041	229	52441
30	900	80	6400	130	16900	180	32400	230	52900
31	961	81	6561	131	17161	181	32761	231	53361
32	1024	82	6724	132	17424	182	33124	232	53824
33	1089	83	6889	133	17689	183	33489	233	54289
34	1156	84	7056	134	17956	184	33856	234	54756
35	1225	85	7225	135	18225	185	34225	235	55225
36	1296	86	7396	136	18496	186	34596	236	55696
37	1369	87	7569	137	18769	187	34969	237	56169
38	1444	88	7744	138	19044	188	35344	238	56644
39	1521	89	7921	139	19321	189	35721	239	57121
40	1600	90	8100	140	19600	190	36100	240	57600
41	1681	91	8281	141	19881	191	36481	241	58081
42	1764	92	8464	142	20164	192	36864	242	58564
43	1849	93	8649	143	20449	193	37249	243	59049
44	1936	94	8836	144	20736	194	37636	244	59536
45	2025	95	9025	145	21025	195	38025	245	60025
46	2116	96	9216	146	21316	196	38416	246	60516
47	2209	97	9409	147	21609	197	38809	247	61009
48	2304	98	9604	148	21904	198	39204	248	61504
49	2401	99	9801	149	22201	199	39601	249	62001
50	2500	100	10000	150	22500	200	40000	250	62500

TABLE 17.1. (*continued*). SQUARES OF NATURAL NUMBERS

n	n^2	n	n^2	n	n^2	n	n^2	n	n^2
251	63001	301	90601	351	123201	401	160801	451	203401
252	63504	302	91204	352	123904	402	161604	452	204304
253	64009	303	91809	353	124609	403	162409	453	205209
254	64516	304	92416	354	125316	404	163216	454	206116
255	65025	305	93025	355	126025	405	164025	455	207025
256	65536	306	93636	356	126736	406	164836	456	207936
257	66049	307	94249	357	127449	407	165649	457	208849
258	66564	308	94864	358	128164	408	166464	458	209764
259	67081	309	95481	359	128881	409	167281	459	210681
260	67600	310	96100	360	129600	410	168100	460	211600
261	68121	311	96721	361	130321	411	168921	461	212521
262	68644	312	97344	362	131044	412	169744	462	213444
263	69169	313	97969	363	131769	413	170569	463	214369
264	69696	314	98596	364	132496	414	171396	464	215296
265	70225	315	99225	365	133225	415	172225	465	216225
266	70756	316	99856	366	133956	416	173056	466	217156
267	71289	317	100489	367	134689	417	173889	467	218089
268	71824	318	101124	368	135424	418	174724	468	219024
269	72361	319	101761	369	136161	419	175561	469	219961
270	72900	320	102400	370	136900	420	176400	470	220900
271	73441	321	103041	371	137641	421	177241	471	221841
272	73984	322	103684	372	138384	422	178084	472	222784
273	74529	323	104329	373	139129	423	178929	473	223729
274	75076	324	104976	374	139876	424	179776	474	224676
275	75625	325	105625	375	140625	425	180625	475	225625
276	76176	326	106276	376	141376	426	181476	476	226576
277	76729	327	106929	377	142129	427	182329	477	227529
278	77284	328	107584	378	142884	428	183184	478	228484
279	77841	329	108241	379	143641	429	184041	479	229441
280	78400	330	108900	380	144400	430	184900	480	230400
281	78961	331	109561	381	145161	431	185761	481	231361
282	79524	332	110224	382	145924	432	186624	482	232324
283	80089	333	110889	383	146689	433	187489	483	233289
284	80656	334	111556	384	147456	434	188356	484	234256
285	81225	335	112225	385	148225	435	189225	485	235225
286	81796	336	112896	386	148996	436	190096	486	236196
287	82369	337	113569	387	149769	437	190969	487	237169
288	82944	338	114244	388	150544	438	191844	488	238144
289	83521	339	114921	389	151321	439	192721	489	239121
290	84100	340	115600	390	152100	440	193600	490	240100
291	84681	341	116281	391	152881	441	194481	491	241081
292	85264	342	116964	392	153664	442	195364	492	242064
293	85849	343	117649	393	154449	443	196249	493	243049
294	86436	344	118336	394	155236	444	197136	494	244036
295	87025	345	119025	395	156025	445	198025	495	245025
296	87616	346	119716	396	156816	446	198916	496	246016
297	88209	347	120409	397	157609	447	199809	497	247009
298	88804	348	121104	398	158404	448	200704	498	248004
299	89401	349	121801	399	159201	449	201601	499	249001
300	90000	350	122500	400	160000	450	202500	500	250000

TABLE 17.1. *(continued)*. SQUARES OF NATURAL NUMBERS

n	n^2	n	n^2	n	n^2	n	n^2	n	n^2
501	251001	551	303601	601	361201	651	423801	701	491401
502	252004	552	304704	602	362404	652	425104	702	492804
503	253009	553	305809	603	363609	653	426409	703	494209
504	254016	554	306916	604	364816	654	427716	704	495616
505	255025	555	308025	605	366025	655	429025	705	497025
506	256036	556	309136	606	367236	656	430336	706	498436
507	257049	557	310249	607	368449	657	431649	707	499849
508	258064	558	311364	608	369664	658	432964	708	501264
509	259081	559	312481	609	370881	659	434281	709	502681
510	260100	560	313600	610	372100	660	435600	710	504100
511	261121	561	314721	611	373321	661	436921	711	505521
512	262144	562	315844	612	374544	662	438244	712	506944
513	263169	563	316969	613	375769	663	439569	713	508369
514	264196	564	318096	614	376996	664	440896	714	509796
515	265225	565	319225	615	378225	665	442225	715	511225
516	266256	566	320356	616	379456	666	443556	716	512656
517	267289	567	321489	617	380689	667	444889	717	514089
518	268324	568	322624	618	381924	668	446224	718	515524
519	269361	569	323761	619	383161	669	447561	719	516961
520	270400	570	324900	620	384400	670	448900	720	518400
521	271441	571	326041	621	385641	671	450241	721	519841
522	272484	572	327184	622	386884	672	451584	722	521284
523	273529	573	328329	623	388129	673	452929	723	522729
524	274576	574	329476	624	389376	674	454276	724	524176
525	275625	575	330625	625	390625	675	455625	725	525625
526	276676	576	331776	626	391876	676	456976	726	527076
527	277729	577	332929	627	393129	677	458329	727	528529
528	278784	578	334084	628	394384	678	459684	728	529984
529	279841	579	335241	629	395641	679	461041	729	531441
530	280900	580	336400	630	396900	680	462400	730	532900
531	281961	581	337561	631	398161	681	463761	731	534361
532	283024	582	338724	632	399424	682	465124	732	535824
533	284089	583	339889	633	400689	683	466489	733	537289
534	285156	584	341056	634	401956	684	467856	734	538756
535	286225	585	342225	635	403225	685	469225	735	540225
536	287296	586	343396	636	404496	686	470596	736	541696
537	288369	587	344569	637	405769	687	471969	737	543169
538	289444	588	345744	638	407044	688	473344	738	544644
539	290521	589	346921	639	408321	689	474721	739	546121
540	291600	590	348100	640	409600	690	476100	740	547600
541	292681	591	349281	641	410881	691	477481	741	549081
542	293764	592	350464	642	412164	692	478864	742	550564
543	294849	593	351649	643	413449	693	480249	743	552049
544	295936	594	352836	644	414736	694	481636	744	553536
545	297025	595	354025	645	416025	695	483025	745	555025
546	298116	596	355216	646	417316	696	484416	746	556516
547	299209	597	356409	647	418609	697	485809	747	558009
548	300304	598	357604	648	419904	698	487204	748	559504
549	301401	599	358801	649	421201	699	488601	749	561001
550	302500	600	360000	650	422500	700	490000	750	562500

TABLE 17.1. *(continued)*. SQUARES OF NATURAL NUMBERS

n	n^2	n	n^2	n	n^2	n	n^2	n	n^2
751	564001	801	641601	851	724201	901	811801	951	904401
752	565504	802	643204	852	725904	902	813604	952	906304
753	567009	803	644809	853	727609	903	815409	953	908209
754	568516	804	646416	854	729316	904	817216	954	910116
755	570025	805	648025	855	731025	905	819025	955	912025
756	571536	806	649636	856	732736	906	820836	956	913936
757	573049	807	651249	857	734449	907	822649	957	915849
758	574564	808	652864	858	736164	908	824464	958	917764
759	576081	809	654481	859	737881	909	826281	959	919681
760	577600	810	656100	860	739600	910	828100	960	921600
761	579121	811	657721	861	741321	911	829921	961	923521
762	580644	812	659344	862	743044	912	831744	962	925444
763	582169	813	660969	863	744769	913	833569	963	927369
764	583696	814	662596	864	746496	914	835396	964	929296
765	585225	815	664225	865	748225	915	837225	965	931225
766	586756	816	665856	866	749956	916	839056	966	933156
767	588289	817	667489	867	751689	917	840889	967	935089
768	589824	818	669124	868	753424	918	842724	968	937024
769	591361	819	670761	869	755161	919	844561	969	938961
770	592900	820	672400	870	756900	920	846400	970	940900
771	594441	821	674041	871	758641	921	848241	971	942841
772	595984	822	675684	872	760384	922	850084	972	944784
773	597529	823	677329	873	762129	923	851929	973	946729
774	599076	824	678976	874	763876	924	853776	974	948676
775	600625	825	680625	875	765625	925	855625	975	950625
776	602176	826	682276	876	767376	926	857476	976	952576
777	603729	827	683929	877	769129	927	859329	977	954529
778	605284	828	685584	878	770884	928	861184	978	956484
779	606841	829	687241	879	772641	929	863041	979	958441
780	608400	830	688900	880	774400	930	864900	980	960400
781	609961	831	690561	881	776161	931	866761	981	962361
782	611524	832	692224	882	777924	932	868624	982	964324
783	613089	833	693889	883	779689	933	870489	983	966289
784	614656	834	695556	884	781456	934	872356	984	968256
785	616225	835	697225	885	783225	935	874225	985	970225
786	617796	836	698896	886	784996	936	876096	986	972196
787	619369	837	700569	887	786769	937	877969	987	974169
788	620944	838	702244	888	788544	938	879844	988	976144
789	622521	839	703921	889	790321	939	881721	989	978121
790	624100	840	705600	890	792100	940	883600	990	980100
791	625681	841	707281	891	793881	941	885481	991	982081
792	627264	842	708964	892	795664	942	887364	992	984064
793	628849	843	710649	893	797449	943	889249	993	986049
794	630436	844	712336	894	799236	944	891136	994	988036
795	632025	845	714025	895	801025	945	893025	995	990025
796	633616	846	715716	896	802816	946	894916	996	992016
797	635209	847	717409	897	804609	947	896809	997	994009
798	636804	848	719104	898	806404	948	898704	998	996004
799	638401	849	720801	899	808201	949	900601	999	998001
800	640000	850	722500	900	810000	950	902500		

TABLE 17.2. SQUARE ROOTS AND THEIR RECIPROCALS

n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$
1	1.0000000	3.1622777	1.0000000	.3162278	51	7.1414284	22.5831796	.1400280	.0442807
2	1.4142136	4.4721360	.7071068	.2236068	52	7.2111026	22.8035085	.1386750	.0438529
3	1.7320508	5.4772256	.5773503	.1825742	53	7.2801099	23.0217289	.1373606	.0434372
4	2.0000000	6.3245553	.5000000	.1581139	54	7.3484692	23.2379001	.1360828	.0430331
5	2.2360680	7.0710678	.4472136	.1414214	55	7.4161985	23.4520788	.1348400	.0426401
6	2.4494897	7.7459667	.4082483	.1290994	56	7.4833148	23.6643191	.1336306	.0422577
7	2.6457513	8.3666003	.3779645	.1195229	57	7.5498344	23.8746728	.1324532	.0418854
8	2.8284271	8.9442719	.3535534	.1118034	58	7.6157731	24.0831892	.1313064	.0415227
9	3.0000000	9.4868330	.3333333	.1054093	59	7.6811457	24.2899156	.1301889	.0411693
10	3.1622777	10.0000000	.3162278	.1000000	60	7.7459667	24.4948974	.1290994	.0408248
11	3.3166248	10.4880885	.3015113	.0953463	61	7.8102497	24.6981781	.1280369	.0404888
12	3.4641016	10.9544512	.2886751	.0912871	62	7.8740079	24.8997992	.1270001	.0401610
13	3.6055513	11.4017543	.2773501	.0877058	63	7.9372539	25.0998008	.1259882	.0398410
14	3.7416574	11.8321596	.2672612	.0845154	64	8.0000000	25.2982213	.1250000	.0395285
15	3.8729833	12.2474487	.2581989	.0816497	65	8.0622577	25.4950976	.1240347	.0392232
16	4.0000000	12.6491106	.2500000	.0790569	66	8.1240384	25.6904652	.1230915	.0389249
17	4.1231056	13.0384048	.2425356	.0766965	67	8.1853528	25.8843582	.1221694	.0386334
18	4.2426407	13.4164079	.2357023	.0745356	68	8.2462113	26.0768096	.1212678	.0383482
19	4.3588989	13.7840488	.2294157	.0725476	69	8.3066239	26.2678511	.1203859	.0380693
20	4.4721360	14.1421356	.2236068	.0707107	70	8.3666003	26.4575131	.1195229	.0377964
21	4.5825757	14.4913767	.2182179	.0690066	71	8.4261498	26.6458252	.1186782	.0375293
22	4.6904158	14.8323970	.2132007	.0674200	72	8.4852814	26.8328157	.1178511	.0372678
23	4.7958315	15.1657509	.2085144	.0659380	73	8.5440037	27.0185122	.1170411	.0370117
24	4.8989795	15.4919334	.2041241	.0645497	74	8.6023253	27.2029410	.1162476	.0367607
25	5.0000000	15.8113883	.2000000	.0632456	75	8.6602540	27.3861279	.1154701	.0365148
26	5.0990195	16.1245155	.1961161	.0620174	76	8.7177979	27.5680975	.1147079	.0362738
27	5.1961524	16.4316767	.1924501	.0608581	77	8.7749644	27.7488739	.1139606	.0360375
28	5.2915026	16.7332005	.1889822	.0597614	78	8.8317609	27.9284801	.1132277	.0358057
29	5.3851648	17.0293864	.1856953	.0587220	79	8.8881944	28.1069386	.1125088	.0355784
30	5.4772256	17.3205081	.1825742	.0577350	80	8.9442719	28.2842712	.1118034	.0353553
31	5.5677644	17.6068169	.1796053	.0567962	81	9.0000000	28.4604989	.1111111	.0351364
32	5.6568542	17.8885438	.1767767	.0559017	82	9.0553851	28.6356421	.1104315	.0349215
33	5.7445626	18.1659021	.1740777	.0550482	83	9.1104336	28.8097206	.1097643	.0347105
34	5.8309519	18.4390889	.1714986	.0542326	84	9.1651514	28.9827535	.1091089	.0345033
35	5.9160798	18.7082869	.1690309	.0534522	85	9.2195445	29.1547595	.1084652	.0342997
36	6.0000000	18.9736660	.1666667	.0527046	86	9.2736185	29.3257566	.1078328	.0340997
37	6.0827625	19.2353841	.1643990	.0519875	87	9.3273791	29.4957624	.1072113	.0339032
38	6.1644140	19.4935887	.1622214	.0512989	88	9.3808315	29.6647939	.1066004	.0337100
39	6.2449980	19.7484177	.1601282	.0506370	89	9.4339811	29.8328678	.1059998	.0335201
40	6.3245553	20.0000000	.1581139	.0500000	90	9.4868330	30.0000000	.1054093	.0333333
41	6.4031242	20.2484567	.1561738	.0493865	91	9.5393920	30.1662063	.1048285	.0331497
42	6.4807407	20.4939015	.1543034	.0487950	92	9.5916630	30.3315018	.1042572	.0329690
43	6.5574385	20.7364414	.1524986	.0482243	93	9.6436508	30.4959014	.1036952	.0327913
44	6.6332496	20.9761770	.1507557	.0476731	94	9.6953597	30.6594194	.1031421	.0326164
45	6.7082039	21.2132034	.1490712	.0471405	95	9.7467943	30.8220700	.1025978	0324443
46	6.7823300	21.4476106	.1474420	.0466252	96	9.7979590	30.9838668	.1020621	.0322749
47	6.8556546	21.6794834	.1458650	.0461266	97	9.8488578	31.1448230	.1015346	.0321081
48	6.9282032	21.9089023	.1443376	.0456435	98	9.8994949	31.3049517	.1010153	.0319438
49	7.0000000	22.1359436	.1428571	.0451754	99	9.9498744	31.4642654	.1005038	.0317821
50	7.0710678	22.3606798	.1414214	.0447214	100	10.0000000	31.6227766	.1000000	.0316228

TABLE 17.2. (*continued*). SQUARE ROOTS AND THEIR RECIPROCALS

n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$
101	10.0498756	31.780497	.0995037	.0314658	151	12.2882057	38.858718	.0813788	.0257343
102	10.0995049	31.937439	.0990148	.0313112	152	12.3288280	38.987177	.0811107	.0256495
103	10.1488916	32.093613	.0985329	.0311588	153	12.3693169	39.115214	.0808452	.0255655
104	10.1980390	32.249031	.0980581	.0310087	154	12.4096736	39.242834	.0805823	.0254824
105	10.2469508	32.403703	.0975900	.0308607	155	12.4498996	39.370039	.0803219	.0254000
106	10.2956301	32.557641	.0971286	.0307148	156	12.4899960	39.496835	.0800641	.0253185
107	10.3440804	32.710854	.0966736	.0305709	157	12.5299641	39.623226	.0798087	.0252377
108	10.3923048	32.863353	.0962250	.0304290	158	12.5698051	39.749214	.0795557	.0251577
109	10.4403065	33.015148	.0957826	.0302891	159	12.6095202	39.874804	.0793052	.0250785
110	10.4880885	33.166248	.0953463	.0301511	160	12.6491106	40.000000	.0790569	.0250000
111	10.5356538	33.316662	.0949158	.0300150	161	12.6885775	40.124805	.0788110	.0249222
112	10.5830052	33.466401	.0944911	.0298807	162	12.7279221	40.249224	.0785674	.0248452
113	10.6301458	33.615473	.0940721	.0297482	163	12.7671453	40.373258	.0783260	.0247689
114	10.6770783	33.763886	.0936586	.0296174	164	12.8062485	40.496913	.0780869	.0246932
115	10.7238053	33.911650	.0932505	.0294884	165	12.8452326	40.620192	.0778499	.0246183
116	10.7703296	34.058773	.0928477	.0293610	166	12.8840987	40.743098	.0776151	.0245440
117	10.8166538	34.205263	.0924500	.0292353	167	12.9228480	40.865633	.0773823	.0244704
118	10.8627805	34.351128	.0920575	.0291111	168	12.9614814	40.987803	.0771517	.0243975
119	10.9087121	34.496377	.0916698	.0289886	169	13.0000000	41.109610	.0769231	.0243252
120	10.9544512	34.641016	.0912871	.0288675	170	13.0384048	41.231056	.0766965	.0242536
121	11.0000000	34.785054	.0909091	.0287480	171	13.0766968	41.352146	.0764719	.0241825
122	11.0453610	34.928498	.0905357	.0286299	172	13.1148770	41.472883	.0762493	.0241121
123	11.0905365	35.071356	.0901670	.0285133	173	13.1529464	41.593269	.0760286	.0240424
124	11.1355287	35.213634	.0898027	.0283981	174	13.1909060	41.713307	.0758098	.0239732
125	11.1803399	35.355339	.0894427	.0282843	175	13.2287566	41.833001	.0755929	.0239046
126	11.2249722	35.496479	.0890871	.0281718	176	13.2664992	41.952354	.0753778	.0238366
127	11.2694277	35.637059	.0887357	.0280607	177	13.3041347	42.071368	.0751646	.0237691
128	11.3137085	35.777088	.0883883	.0279508	178	13.3416641	42.190046	.0749532	.0237023
129	11.3578167	35.916570	.0880451	.0278423	179	13.3790882	42.308392	.0747435	.0236360
130	11.4017543	36.055513	.0877058	.0277350	180	13.4164079	42.426407	.0745356	.0235702
131	11.4455231	36.193922	.0873704	.0276289	181	13.4536240	42.544095	.0743294	.0235050
132	11.4891253	36.331804	.0870388	.0275241	182	13.4907376	42.661458	.0741249	.0234404
133	11.5325626	36.469165	.0867110	.0274204	183	13.5277493	42.778499	.0739221	.0233762
134	11.5758369	36.606010	.0863868	.0273179	184	13.5646600	42.895221	.0737210	.0233126
135	11.6189500	36.742346	.0860663	.0272166	185	13.6014705	43.011626	.0735215	.0232495
136	11.6619038	36.878178	.0857493	.0271163	186	13.6381817	43.127717	.0733236	.0231869
137	11.7046999	37.013511	.0854358	.0270172	187	13.6747943	43.243497	.0731272	.0231249
138	11.7473401	37.148351	.0851257	.0269191	188	13.7113092	43.358967	.0729325	.0230633
139	11.7898261	37.282704	.0848189	.0268221	189	13.7477271	43.474130	.0727393	.0230022
140	11.8321596	37.416574	.0845154	.0267261	190	13.7840488	43.588989	.0725476	.0229416
141	11.8743421	37.549967	.0842152	.0266312	191	13.8202750	43.703547	.0723575	.0228814
142	11.9163753	37.682887	.0839181	.0265372	192	13.8564065	43.817805	.0721688	.0228218
143	11.9582607	37.815341	.0836242	.0264443	193	13.8924440	43.931765	.0719816	.0227626
144	12.0000000	37.947332	.0833333	.0263523	194	13.9283883	44.045431	.0717958	.0227038
145	12.0415946	38.078866	.0830455	.0262613	195	13.9642400	44.158804	.0716115	.0226455
146	12.0830460	38.209946	.0827606	.0261712	196	14.0000000	44.271887	.0714286	.0225877
147	12.1243557	38.340579	.0824786	.0260820	197	14.0356688	44.384682	.0712470	.0225303
148	12.1655251	38.470768	.0821995	.0259938	198	14.0712473	44.497191	.0710669	.0224733
149	12.2065556	38.600518	.0819232	.0259064	199	14.1067360	44.609416	.0708881	.0224168
150	12.2474487	38.729833	.0816497	.0258199	200	14.1421356	44.721360	.0707107	.0223607

TABLE 17.2. (*continued*). SQUARE ROOTS AND THEIR RECIPROCALS

n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$
201	14.1774469	44.833024	.0705346	.0223050	251	15.8429795	50.099900	.0631194	.0199601
202	14.2126704	44.944410	.0703598	.0222497	252	15.8745079	50.199602	.0629941	.0199205
203	14.2478068	45.055521	.0701862	.0221948	253	15.9059737	50.299105	.0628695	.0198811
204	14.2828569	45.166359	.0700140	.0221404	254	15.9373775	50.398413	.0627456	.0198419
205	14.3178211	45.276926	.0698430	.0220863	255	15.9687194	50.497525	.0626224	.0198030
206	14.3527001	45.387223	.0696733	.0220326	256	16.0000000	50.596443	.0625000	.0197642
207	14.3874946	45.497253	.0695048	.0219793	257	16.0312195	50.695167	.0623783	.0197257
208	14.4222051	45.607017	.0693375	.0219265	258	16.0623784	50.793700	.0622573	.0196875
209	14.4568323	45.716518	.0691714	.0218739	259	16.0934769	50.892043	.0621370	.0196494
210	14.4913767	45.825757	.0690066	.0218218	260	16.1245155	50.990195	.0620174	.0196116
211	14.5258390	45.934736	.0688428	.0217700	261	16.1554944	51.088159	.0618984	.0195740
212	14.5602198	46.043458	.0686803	.0217186	262	16.1864141	51.185936	.0617802	.0195366
213	14.5945195	46.151923	.0685189	.0216676	263	16.2172747	51.283526	.0616626	.0194994
214	14.6287388	46.260134	.0683586	.0216169	264	16.2480768	51.380930	.0615457	.0194625
215	14.6628783	46.368092	.0681994	.0215666	265	16.2788206	51.478151	.0614295	.0194257
216	14.6969385	46.475800	.0680414	.0215166	266	16.3095064	51.575188	.0613139	.0193892
217	14.7309199	46.583259	.0678844	.0214669	267	16.3401346	51.672043	.0611990	.0193528
218	14.7648231	46.690470	.0677285	.0214176	268	16.3707055	51.768716	.0610847	.0193167
219	14.7986486	46.797436	.0675737	.0213687	269	16.4012195	51.865210	.0609711	.0192807
220	14.8323970	46.904158	.0674200	.0213201	270	16.4316767	51.961524	.0608581	.0192450
221	14.8660687	47.010637	.0672673	.0212718	271	16.4620776	52.057660	.0607457	.0192095
222	14.8996644	47.116876	.0671156	.0212238	272	16.4924225	52.153619	.0606339	.0191741
223	14.9331845	47.222876	.0669650	.0211762	273	16.5227116	52.249402	.0605228	.0191390
224	14.9666295	47.328638	.0668153	.0211289	274	16.5529454	52.345009	.0604122	.0191040
225	15.0000000	47.434165	.0666667	.0210819	275	16.5831240	52.440442	.0603023	.0190693
226	15.0332964	47.539457	.0665190	.0210352	276	16.6132477	52.535702	.0601929	.0190347
227	15.0665192	47.644517	.0663723	.0209888	277	16.6433170	52.630789	.0600842	.0190003
228	15.0996689	47.749346	.0662266	.0209427	278	16.6733320	52.725705	.0599760	.0189661
229	15.1327460	47.853944	.0660819	.0208969	279	16.7032931	52.820451	.0598684	.0189321
230	15.1657509	47.958315	.0659380	.0208514	280	16.7332005	52.915026	.0597614	.0188982
231	15.1986842	48.062459	.0657952	.0208063	281	16.7630546	53.009433	.0596550	.0188646
232	15.2315462	48.166378	.0656532	.0207614	282	16.7928556	53.103672	.0595491	.0188311
233	15.2643375	48.270074	.0655122	.0207168	283	16.8226038	53.197744	.0594438	.0187978
234	15.2970585	48.373546	.0653720	.0206725	284	16.8522995	53.291650	.0593391	.0187647
235	15.3297097	48.476799	.0652328	.0206284	285	16.8819430	53.385391	.0592349	.0187317
236	15.3622915	48.579831	.0650945	.0205847	286	16.9115345	53.478968	.0591312	.0186989
237	15.3948043	48.682646	.0649570	.0205412	287	16.9410743	53.572381	.0590281	.0186663
238	15.4272486	48.785244	.0648204	.0204980	288	16.9705627	53.665631	.0589256	.0186339
239	15.4596248	48.887626	.0646846	.0204551	289	17.0000000	53.758720	.0588235	.0186016
240	15.4919334	48.989795	.0645497	.0204124	290	17.0293864	53.851648	.0587220	.0185695
241	15.5241747	49.091751	.0644157	.0203700	291	17.0587221	53.944416	.0586210	.0185376
242	15.5563492	49.193496	.0642824	.0203279	292	17.0880075	54.037024	.0585206	.0185058
243	15.5884573	49.295030	.0641500	.0202860	293	17.1172428	54.129474	.0584206	.0184742
244	15.6204994	49.396356	.0640184	.0202444	294	17.1464282	54.221767	.0583212	.0184428
245	15.6524758	49.497475	.0638877	.0202031	295	17.1755640	54.313902	.0582223	.0184115
246	15.6843871	49.598387	.0637577	.0201619	296	17.2046505	54.405882	.0581238	.0183804
247	15.7162336	49.699095	.0636285	.0201211	297	17.2336879	54.497706	.0580259	.0183494
248	15.7480157	49.799598	.0635001	.0200805	298	17.2626765	54.589376	.0579284	.0183186
249	15.7797338	49.899900	.0633724	.0200401	299	17.2916165	54.680892	.0578315	.0182879
250	15.8113883	50.000000	.0632456	.0200000	300	17.3205081	54.772256	.0577350	.0182574

TABLE 17.2. (*continued*). SQUARE ROOTS AND THEIR RECIPROCALS

n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	$1/\sqrt{n}$	$1/\sqrt{10n}$
301	17.3493516	54.863467	.0576390	.0182271	351	18.7349940	59.245253	.0533761	.0168790
302	17.3781472	54.954527	.0575435	.0181969	352	18.7616630	59.329588	.0533002	.0168550
303	17.4068952	55.045436	.0574485	.0181668	353	18.7882942	59.413803	.0532246	.0168311
304	17.4355958	55.136195	.0573539	.0181369	354	18.8148877	59.497899	.0531494	.0168073
305	17.4642492	55.226805	.0572598	.0181071	355	18.8414437	59.581876	.0530745	.0167836
306	17.4928557	55.317267	.0571662	.0180775	356	18.8679623	59.665736	.0529999	.0167600
307	17.5214155	55.407581	.0570730	.0180481	357	18.8944436	59.749477	.0529256	.0167365
308	17.5499288	55.497748	.0569803	.0180187	358	18.9208879	59.833101	.0528516	.0167132
309	17.5783958	55.587768	.0568880	.0179896	359	18.9472953	59.916609	.0527780	.0166899
310	17.6068169	55.677644	.0567962	.0179605	360	18.9736660	60.000000	.0527046	.0166667
311	17.6351921	55.767374	.0567048	.0179316	361	19.0000000	60.083276	.0526316	.0166436
312	17.6635217	55.856960	.0566139	.0179029	362	19.0262976	60.166436	.0525588	.0166206
313	17.6918060	55.946403	.0565233	.0178743	363	19.0525589	60.249481	.0524864	.0165977
314	17.7200451	56.035703	.0564333	.0178458	364	19.0787840	60.332413	.0524142	.0165748
315	17.7482393	56.124861	.0563436	.0178174	365	19.1049732	60.415230	.0523424	.0165521
316	17.7763888	56.213877	.0562544	.0177892	366	19.1311265	60.497934	.0522708	.0165295
317	17.8044938	56.302753	.0561656	.0177611	367	19.1572441	60.580525	.0521996	.0165070
318	17.8325545	56.391489	.0560772	.0177332	368	19.1833261	60.663004	.0521286	.0164845
319	17.8605711	56.480085	.0559893	.0177054	369	19.2093727	60.745370	.0520579	.0164622
320	17.8885438	56.568542	.0559017	.0176777	370	19.2353841	60.827625	.0519875	.0164399
321	17.9164729	56.656862	.0558146	.0176501	371	19.2613603	60.909769	.0519174	.0164177
322	17.9443584	56.745044	.0557278	.0176227	372	19.2873015	60.991803	.0518476	.0163956
323	17.9722008	56.833089	.0556415	.0175954	373	19.3132079	61.073726	.0517780	.0163737
324	18.0000000	56.920998	.0555556	.0175682	374	19.3390796	61.155539	.0517088	.0163517
325	18.0277564	57.008771	.0554700	.0175412	375	19.3649167	61.237244	.0516398	.0163299
326	18.0554701	57.096410	.0553849	.0175142	376	19.3907194	61.318839	.0515711	.0163082
327	18.0831413	57.183914	.0553001	.0174874	377	19.4164878	61.400326	.0515026	.0162866
328	18.1107703	57.271284	.0552158	.0174608	378	19.4422221	61.481705	.0514345	.0162650
329	18.1383571	57.358522	.0551318	.0174342	379	19.4679223	61.562976	.0513665	.0162435
330	18.1659021	57.445626	.0550482	.0174078	380	19.4935887	61.644140	.0512989	.0162221
331	18.1934054	57.532599	.0549650	.0173814	381	19.5192213	61.725197	.0512316	.0162008
332	18.2208672	57.619441	.0548821	.0173553	382	19.5448203	61.806149	.0511645	.0161796
333	18.2482876	57.706152	.0547997	.0173292	383	19.5703858	61.886994	.0510976	.0161585
334	18.2756669	57.792733	.0547176	.0173032	384	19.5959179	61.967734	.0510310	.0161374
335	18.3030052	57.879185	.0546358	.0172774	385	19.6214169	62.048368	.0509647	.0161165
336	18.3303028	57.965507	.0545545	.0172516	386	19.6468827	62.128898	.0508987	.0160956
337	18.3575598	58.051701	.0544735	.0172260	387	19.6723156	62.209324	.0508329	.0160748
338	18.3847763	58.137767	.0543928	.0172005	388	19.6977156	62.289646	.0507673	.0160540
339	18.4119526	58.223707	.0543125	.0171751	389	19.7230829	62.369865	.0507020	.0160334
340	18.4390889	58.309519	.0542326	.0171499	390	19.7484177	62.449980	.0506370	.0160128
341	18.4661853	58.395205	.0541530	.0171247	391	19.7737199	62.529993	.0505722	.0159923
342	18.4932420	58.480766	.0540738	.0170996	392	19.7989899	62.609903	.0505076	.0159719
343	18.5202592	58.566202	.0539949	.0170747	393	19.8242276	62.689712	.0504433	.0159516
344	18.5472370	58.651513	.0539164	.0170499	394	19.8494332	62.769419	.0503793	.0159313
345	18.5741756	58.736701	.0538382	.0170251	395	19.8746069	62.849025	.0503155	.0159111
346	18.6010752	58.821765	.0537603	.0170005	396	19.8997487	62.928531	.0502519	.0158910
347	18.6279360	58.906706	.0536828	.0169760	397	19.9248588	63.007936	.0501886	.0158710
348	18.6547581	58.991525	.0536056	.0169516	398	19.9499373	63.087241	.0501255	.0158511
349	18.6815417	59.076222	.0535288	.0169273	399	19.9749844	63.166447	.0500626	.0158312
350	18.7082869	59.160798	.0534522	.0169031	400	20.0000000	63.245553	.0500000	.0158114

TABLE 17.3. CUBES AND CUBEROOTS, FOURTH POWERS AND FOURTH ROOTS, RECIPROCALS, FACTORIALS, EXPONENTIALS AND NATURAL LOGARITHMS

n	n^3	n^4	$n!$	$\log_e n$	e^{-n}	$e^{-n/100}$	$\sqrt[3]{n}$	$\sqrt[4]{n}$	$1/n$	n
1	1	1	1	0.000000	.367879	.990050	1.0000000	1.000000	1.000000000	1
2	8	16	2	0.693147	.135335	.980199	1.2599210	1.189207	.500000000	2
3	27	81	6	1.098612	.049787	.970446	1.4422496	1.316074	.333333333	3
4	64	256	24	1.386294	.018316	.960789	1.5874011	1.414214	.250000000	4
5	125	625	120	1.609438	.673795 (2)	.951229	1.7099759	1.495349	.200000000	5
6	216	1296	720	1.791759	.247875 (2)	.941765	1.8171206	1.565085	.166666667	6
7	343	2401	5040	1.945910	.911882 (3)	.932394	1.9129312	1.626577	.142857143	7
8	512	4096	40320	2.079442	.335463 (3)	.923116	2.0000000	1.681793	.125000000	8
9	729	6561	362880	2.197225	.123410 (3)	.913931	2.0800838	1.732051	.111111111	9
10	1000	10000	0.3628800 (7)	2.302585	.453999 (4)	.904837	2.1544347	1.778279	.100000000	10
11	1331	14641	3.9916800 (7)	2.397895	.167017 (4)	.895834	2.2239801	1.821160	.090909091	11
12	1728	20736	4.7900160 (8)	2.484907	.614421 (5)	.886920	2.2894285	1.861210	.083333333	12
13	2197	28561	6.2270208 (9)	2.564949	.226033 (5)	.878095	2.3513347	1.898829	.076923077	13
14	2744	38416	8.7178291 (10)	2.639057	.831529 (6)	.869358	2.4101423	1.934336	.071428571	14
15	3375	50625	1.3076744 (12)	2.708050	.305902 (6)	.860708	2.4662121	1.967990	.066666667	15
16	4096	65536	2.0922790 (13)	2.772589	.112535 (6)	.852144	2.5198421	2.000000	.062500000	16
17	4913	83521	3.5568743 (14)	2.833213	.413994 (7)	.843665	2.5712816	2.030543	.058823529	17
18	5832	104976	6.4023737 (15)	2.890372	.152300 (7)	.835270	2.6207414	2.059767	.055555556	18
19	6859	130321	1.2164510 (17)	2.944439	.560280 (8)	.826959	2.6684016	2.087798	.052631579	19
20	8000	160000	2.4329020 (18)	2.995732	.206115 (8)	.818731	2.7144176	2.114743	.050000000	20
21	9261	194481	5.1090942 (19)	3.044522	.758256 (9)	.810584	2.7589242	2.140695	.047619048	21
22	10648	234256	1.1240007 (21)	3.091042	.278947 (9)	.802519	2.8020393	2.165737	.045454545	22
23	12167	279841	2.5852017 (22)	3.135494	.102619 (9)	.794534	2.8438670	2.189939	.043478261	23
24	13824	331776	6.2044840 (23)	3.178054	.377513 (10)	.786628	2.8844991	2.213364	.041666667	24
25	15625	390625	1.5511210 (25)	3.218876	.138879 (10)	.778801	2.9240177	2.236068	.040000000	25
26	17576	456976	4.0329146 (26)	3.258097	.510909 (11)	.771052	2.9624961	2.258101	.038461538	26
27	19683	531441	1.0888869 (28)	3.295837	.187953 (11)	.763379	3.0000000	2.279507	.037037037	27
28	21952	614656	3.0488834 (29)	3.332205	.691440 (12)	.755784	3.0365890	2.300327	.035714286	28
29	24389	707281	8.8417620 (30)	3.367296	.254367 (12)	.748264	3.0723168	2.320596	.034482759	29
30	27000	810000	2.6525286 (32)	3.401197	.935762 (13)	.740818	3.1072325	2.340347	.033333333	30
31	29791	923521	8.2228387 (33)	3.433987	.344248 (13)	.733447	3.1413807	2.359611	.032258065	31
32	32768	1048576	2.6313084 (35)	3.465736	.126642 (13)	.726149	3.1748021	2.378414	.031250000	32
33	35937	1185921	8.6833176 (36)	3.496508	.465889 (14)	.718924	3.2075343	2.396782	.030303030	33
34	39304	1336336	2.9523280 (38)	3.526361	.171391 (14)	.711770	3.2396118	2.414736	.029411765	34
35	42875	1500625	1.0333148 (40)	3.555348	.630512 (15)	.704688	3.2710663	2.432299	.028571429	35

The number in brackets following $n!$ is the power of 10 and that following e^{-n} is the power of 1/10 by which the given tabular value must be multiplied.

TABLE 17.3. (*continued*). CUBES AND CUBEROOTS, FOURTH POWERS AND FOURTH ROOTS; RECIPROCALS, FACTORIALS, EXPONENTIALS AND NATURAL LOGARITHMS

n	n^3	n^4	$n!$	$\log_e n$	e^{-n}	$e^{-n/100}$	$\sqrt[3]{n}$	$\sqrt[4]{n}$	$1/n$	n
36	46656	1679616	3.7199333 (41)	3.583519	.231952 (15)	.697676	3.3019272	2.449490	.027777778	36
37	50653	1874161	1.3763753 (43)	3.610918	.853305 (16)	.690734	3.3322219	2.466326	.027027027	37
38	54872	2085136	5.2302262 (44)	3.637586	.313913 (16)	.683861	3.3619754	2.482824	.026315789	38
39	59319	2313441	2.0397882 (46)	3.663562	.115482 (16)	.677057	3.3912114	2.498999	.025641026	39
40	64000	2560000	8.1591528 (47)	3.688879	.424835 (17)	.670320	3.4199519	2.514867	.025000000	40
41	68921	2825761	3.3452527 (49)	3.713572	.156288 (17)	.663650	3.4482172	2.530440	.024390244	41
42	74088	3111696	1.4050061 (51)	3.737670	.574952 (18)	.657047	3.4760266	2.545730	.023809524	42
43	79507	3418801	6.0415263 (52)	3.761200	.211513 (18)	.650509	3.5033981	2.560750	.023255814	43
44	85184	3748096	2.6582716 (54)	3.784190	.778113 (19)	.644036	3.5303483	2.575510	.022727273	44
45	91125	4100625	1.1962222 (56)	3.806662	.286252 (19)	.637628	3.5568933	2.590020	.022222222	45
46	97336	4477456	5.5026222 (57)	3.828641	.105306 (19)	.631284	3.5830479	2.604291	.021739130	46
47	103823	4879681	2.5862324 (59)	3.850148	.387400 (20)	.625002	3.6088261	2.618330	.021276596	47
48	110592	5308416	1.2413916 (61)	3.871201	.142516 (20)	.618783	3.6342412	2.632148	.020833333	48
49	117649	5764801	6.0828186 (62)	3.891820	.524289 (21)	.612626	3.6593057	2.645751	.020408163	49
50	125000	6250000	3.0414093 (64)	3.912023	.192875 (21)	.606531	3.6840315	2.659148	.020000000	50
51	132651	6765201	1.5511188 (66)	3.931826	.709547 (22)	.600496	3.7084298	2.672345	.019607843	51
52	140608	7311616	8.0658175 (67)	3.951244	.261028 (22)	.594521	3.7325112	2.685350	.019230769	52
53	148877	7890481	4.2748833 (69)	3.970292	.960268 (23)	.588605	3.7562858	2.698168	.018867925	53
54	157464	8503056	2.3084370 (71)	3.988984	.353263 (23)	.582748	3.7797631	2.710806	.018518519	54
55	166375	9150625	1.2696403 (73)	4.007333	.129958 (23)	.576950	3.8029525	2.723270	.018181818	55
56	175616	9834496	7.1099859 (74)	4.025352	.478089 (24)	.571209	3.8258624	2.735565	.017857143	56
57	185193	10556001	4.0526920 (76)	4.043051	.175879 (24)	.565525	3.8485011	2.747696	.017543860	57
58	195112	11316496	2.3505613 (78)	4.060443	.647023 (25)	.559898	3.8708766	2.759669	.017241379	58
59	205379	12117361	1.3868312 (80)	4.077537	.238027 (25)	.554327	3.8929964	2.771488	.016949153	59
60	216000	12960000	8.3209871 (81)	4.094345	.875651 (26)	.548812	3.9148676	2.783158	.016666667	60
61	226981	13845841	5.0758021 (83)	4.110874	.322134 (26)	.543351	3.9364972	2.794682	.016393443	61
62	238328	14776336	3.1469973 (85)	4.127134	.118506 (26)	.537944	3.9578916	2.806066	.016129032	62
63	250047	15752961	1.9823083 (87)	4.143135	.435961 (27)	.532592	3.9790572	2.817313	.015873016	63
64	262144	16777216	1.2688693 (89)	4.158883	.160381 (27)	.527292	4.0000000	2.828427	.015625000	64
65	274625	17850625	8.2476506 (90)	4.174387	.590009 (28)	.522046	4.0207258	2.839412	.015384615	65
66	287496	18974736	5.4434494 (92)	4.189655	.217052 (28)	.516851	4.0412400	2.850270	.015151515	66
67	300763	20151121	3.6471111 (94)	4.204693	.798490 (29)	.511709	4.0615481	2.861006	.014925373	67
68	314432	21381376	2.4800355 (96)	4.219508	.293748 (29)	.506617	4.0816551	2.871622	.014705882	68
69	328509	22667121	1.7112245 (98)	4.234107	.108064 (29)	.501576	4.1015659	2.882121	.014492754	69
70	343000	24010000	1.1978572 (100)	4.248495	.397545 (30)	.496585	4.1212853	2.892508	.014285714	70

The number in brackets following $n!$ is the power of 10 and that following e^{-n} is the power of 1/10 by which the given tabular value must be multiplied.

TABLE 17.3. (*continued*). CUBES AND CUBEROOTS, FOURTH POWERS AND FOURTH ROOTS RECIPROCALS, FACTORIALS, EXPONENTIALS AND NATURAL LOGARITHMS

n	n^3	n^4	$n!$	$\log_e n$	e^{-n}	$e^{-n/100}$	$\sqrt[3]{n}$	$\sqrt[4]{n}$	$1/n$	n
71	357911	25411681	8.5047859 (101)	4.262680	.146249 (30)	.491644	4.1408177	2.902783	.014084507	71
72	373248	26873856	6.1234458 (103)	4.276666	.538019 (31)	.486752	4.1601676	2.912951	.013888889	72
73	389017	28398241	4.4701155 (105)	4.290459	.197926 (31)	.481909	4.1793392	2.923013	.013698630	73
74	405224	29986576	3.3078854 (107)	4.304065	.728129 (32)	.477114	4.1983365	2.932972	.013513514	74
75	421875	31640625	2.4809141 (109)	4.317488	.267864 (32)	.472367	4.2171633	2.942831	.013333333	75
76	438976	33362176	1.8854947 (111)	4.330733	.985415 (33)	.467666	4.2358236	2.952592	.013157895	76
77	456533	35153041	1.4518309 (113)	4.343805	.362514 (33)	.463013	4.2543209	2.962257	.012987013	77
78	474552	37015056	1.1324281 (115)	4.356709	.133361 (33)	.458406	4.2726587	2.971828	.012820513	78
79	493039	38950081	8.9461821 (116)	4.369448	.490609 (34)	.453845	4.2908404	2.981308	.012658228	79
80	512000	40960000	7.1569457 (118)	4.382027	.180485 (34)	.449329	4.3088694	2.990698	.012500000	80
81	531441	43046721	5.7971260 (120)	4.394449	.663968 (35)	.444858	4.3267487	3.000000	.012345679	81
82	551368	45212176	4.7536433 (122)	4.406719	.244260 (35)	.440432	4.3444815	3.009217	.012195122	82
83	571787	47458321	3.9455240 (124)	4.418841	.898583 (36)	.436049	4.3620707	3.018349	.012048193	83
84	592704	49787136	3.3142401 (126)	4.430817	.330570 (36)	.431711	4.3795191	3.027400	.011904762	84
85	614125	52200625	2.8171041 (128)	4.442651	.121610 (36)	.427415	4.3968297	3.036370	.011764706	85
86	636056	54700816	2.4227095 (130)	4.454347	.447378 (37)	.423162	4.4140050	3.045262	.011627907	86
87	658503	57289761	2.1077573 (132)	4.465908	.164581 (37)	.418952	4.4310476	3.054076	.011494253	87
88	681472	59969536	1.8548264 (134)	4.477337	.605460 (38)	.414783	4.4479602	3.062814	.011363636	88
89	704969	62742241	1.6507955 (136)	4.488636	.222736 (38)	.410656	4.4647451	3.071479	.011235955	89
90	729000	65610000	1.4857160 (138)	4.499810	.819401 (39)	.406570	4.4814047	3.080070	.011111111	90
91	753571	68574961	1.3520015 (140)	4.510860	.301441 (39)	.402524	4.4979414	3.088591	.010989011	91
92	778688	71639296	1.2438414 (142)	4.521789	.110894 (39)	.398519	4.5143574	3.097041	.010869565	92
93	804357	74805201	1.1567725 (144)	4.532599	.407956 (40)	.394554	4.5306549	3.105423	.010752688	93
94	830584	78074896	1.0873662 (146)	4.543295	.150079 (40)	.390628	4.5468359	3.113737	.010638298	94
95	857375	81450625	1.0329978 (148)	4.553877	.552108 (41)	.386741	4.5629026	3.121986	.010526316	95
96	884736	84934656	9.9167793 (149)	4.564348	.203109 (41)	.382893	4.5788570	3.130169	.010416667	96
97	912673	88529281	9.6192760 (151)	4.574711	.747197 (42)	.379083	4.5947009	3.138289	.010309278	97
98	941192	92236816	9.4268904 (153)	4.584967	.274879 (42)	.375311	4.6104363	3.146346	.010204082	98
99	970299	96059601	9.3326215 (155)	4.595120	.101122 (42)	.371577	4.6260650	3.154342	.010101010	99
100	1000000	100000000	9.3326215 (157)	4.605170	.372008 (43)	.367879	4.6415888	3.162278	.010000000	100

The number in brackets following $n!$ is the power of 10 and that following e^{-n} is the power of 1/10 by which the given tabular value must be multiplied.

TABLE 17.4. HIGHER POWERS OF NATURAL NUMBERS

n	n^5	n^6	n^7	n^8	n^9	n^{10}	n^{11}
1	1	1	1	1	1	1	1
2	32	64	128	256	512	1024	2048
3	243	729	2187	6561	19683	59049	1 77147
4	1024	4096	16384	65536	2 62144	10 48576	41 94304
5	3125	15625	78125	3 90625	19 53125	97 65625	488 28125
6	7776	46656	2 79936	16 79616	100 77696	604 66176	3627 97056
7	16807	1 17649	8 23543	57 64801	403 53607	2824 75249	19773 26743
8	32768	2 62144	20 97152	167 77216	1342 17728	10737 41824	85899 34592
9	59049	5 31441	47 82969	430 46721	3874 20489	34867 84401	3 13810 59609

n	n^{12}	n^{13}	n^{14}	n^{15}	n^{16}
1	1	1	1	1	1
2	4096	8192	16384	32768	65536
3	5 31441	15 94323	47 82969	143 48907	430 46721
4	167 77216	671 08864	2684 35456	10737 41824	42949 67296
5	2441 40625	12207 03125	61035 15625	3 05175 78125	15 25878 90625
6	21767 82336	1 30606 94016	7 83641 64096	47 01849 84576	282 11099 07456
7	1 38412 87201	9 68890 10407	67 82230 72849	474 75615 09943	3323 29305 69601
8	6 87194 76736	54 97558 13888	439 80465 11104	3518 43720 88832	28147 49767 10656
9	28 24295 36481	254 18658 28329	2287 67924 54961	20589 11320 94649	1 85302 01888 51841

n	n^{17}	n^{18}	n^{19}	n^{20}
1	1	1	1	1
2	1 31072	2 62144	5 24288	10 48576
3	1291 40163	3874 20489	11622 61467	34867 84401
4	1 71798 69184	6 87194 76736	27 48779 06944	109 95116 27776
5	76 29394 53125	381 46972 65625	1907 34863 28125	9536 74316 40625
6	1692 66594 44736	10155 99566 68416	60935 97400 10496	3 65615 84400 62976
7	23263 05139 87207	1 62841 35979 10449	11 39889 51853 73143	79 79226 62976 12001
8	2 25179 98136 85248	18 01439 85094 81984	144 11518 80758 55872	1152 92150 46068 46976
9	16 67718 16996 66569	150 09463 52969 99121	1350 85171 76729 92089	12157 66545 90569 28801

17.5. CONVERSION OF NUMBER SYSTEMS

a. Introduction

Most of the digital computers carry out the arithmetical operations in number systems such as the binary (radix 2), ternary (radix 3), octal (radix 8) and hexadecimal (radix 16). Decimal numbers (radix 10) have, therefore, to be converted to other systems at the stage of input into the machine and the results at the stage of output have to be converted back into the decimal system. Table 17.5 which furnishes positive and negative powers of 2, 3, 8 and 16, is useful for this purpose. The table also gives three digited binary equivalents for numbers 0 to 7 and four digited binary equivalents for numbers 0 to 15.

b. Conversion between the decimal and other systems

Example 1. The number

$$(367.6102)_8$$

in the octal system is equivalent to

$$3\times 8^2+6\times 8+7\times 8^0+6\times 8^{-1}+1\times 8^{-2}+0\times 8^{-3}+2\times 8^{-4}$$
$$=(247.76113281\ \ldots)_{10}$$

in the decimal system. To arrive at this value, the positive and negative powers of 8 have been used from Table 17.5 (powers of eight).

Example 2. To convert $(247.76113)_{10}$ into octal and hexadecimal systems. The integral part 247 and the decimal part .76113 have to be considered separately. To convert the former into the octal system, it is first divided by 8 and the remainder noted, the quotient is then divided by 8 and the remainder again noted; this is continued until the quotient obtained is zero. Thus,

	quotient	remainder
$247 \div 8$	30	7
$30 \div 8$	3	6
$3 \div 8$	0	3

Collecting the remainders,

$$(247)_{10}=(367)_8.$$

As regards the decimal part .76113, repeated multiplication by 8, each time omitting the integer in the unit's place, is carried out as follows :

$$.76113\times 8 = 6.08904$$
$$.08904\times 8 = 0.71232$$
$$71232\times 8 = 5.69856$$
$$\ldots \qquad \ldots$$

yielding $(.76113)_{10}=(.605\ \ldots)_8$. The final answer is obtained by putting the two conversions together. Thus,

$$(247.76113)_{10}=(367.605,..)_8.$$

In the hexadecimal system there are 16 symbols. The symbols 0, 1, ..., 9 may be used for the digits 0, ..., 9 and t, u, v, w, x, y for 10, 11, 12, 13, 14, 15. The conversion of

$(247.76113)_{10}$ is done as follows:

	quotient	remainder
$247 \div 16$	15	7
$15 \div 16$	0	$15 = y$

$$(247)_{10} = (y7)_{16}$$

$$.76113 \times 16 = 12.17808$$

$$.17808 \times 16 = 2.84928$$

$$.84928 \times 16 = 13.58848$$

$$\cdots \qquad \cdots$$

$$(.76113)_{10} = (.v\ 2\ w\ \ldots)_{16}$$

$$(247.76113)_{10} = (y7.v\ 2\ w\ \ldots)_{16}$$

Example 3. To convert $(1000111000)_2$ in the binary system to decimal system. $1 \times 2^9 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 = (568)_{10}$ in the decimal system as obtained by using the powers of 2 given in Table 17.5 (powers of 2). Similarly,

$$(1100.1101)$$
$$= 2^3 + 2^2 + 2^{-1} + 2^{-2} + 2^{-4} = (12.8125)_{10}$$

The conversion from decimal to binary system is done by successive divisions and multiplications by 2 of the integral and decimal parts respectively as in example 2.

b. Conversion between the binary and octal or hexadecimal systems

Example 4. Convert $(1000111000)_2$ into octal system. This is done very easily by breaking the given number into sets of 3 digits and writing down the octal equivalent of each set using Table 17.5 (binary equivalents). When an incomplete set is found at the beginning, zeros are placed to complete it. Thus,

	1,000, 111, 000
is written	001, 000, 111, 000
with octal equivalents	1 0 7 0
giving	$(1{,}000,\ 111,\ 000)_2 = (1070)_8$

The conversion from octal to binary system consists in simply replacing each octal digit by the corresponding triplet of the binary system using Table 17.5. Thus,

$$(1\ 4\ 6)_8 = (001,\ 100,\ 110)_2 = (1100110)_2$$

Example 5. Convert $(1100011.1011)_2$ into octal system. The division into sets is done as follows:

$$1,\ 100,\ 011.\ 101,\ 1$$

starting from the left for digits preceding the binary point and from the right for digits following the binary point. The incomplete sets are completed and the octal equivalents of the sets are written

$$001,\ 100,\ 011.\ 101,\ 100$$
$$1 \quad 4 \quad 3. \quad 5 \quad 4$$

Thus, $(1100011.1011)_2 = (143.54)_8$

Example 6. Convert $(1100011.1011)_2$ into hexadecimal system. The procedure using the equivalents given in Table 17.5 (binary equivalents) is the same as inexample 5.

Such simple methods are not generally available for conversion from one system to another. To convert a number with radix b to one with radix c, a general procedure is to convert the number with radix b to decimal system and then convert it to radix c.

TABLE 17.5. CONVERSION OF NUMBER SYSTEMS

POWERS OF TWO

n	2^n	2^{-n}
0	1	1.0
1	2	0.5
2	4	0.25
3	8	0.125
4	16	0.062 5
5	32	0.031 25
6	64	0.015 625
7	128	0.007 812 5
8	256	0.003 906 25
9	512	0.001 953 125
10	1 024	0.000 976 562 5
11	2 048	0.000 488 281 25
12	4 096	0.000 244 140 625
13	8 192	0.000 122 070 312 5
14	16 384	0.000 061 035 156 25
15	32 768	0.000 030 517 578 125
16	65 536	0.000 015 258 789 062 5
17	131 072	0.000 007 629 394 531 25
18	262 144	0.000 003 814 697 265 625
19	524 288	0.000 001 907 348 632 812 5
20	1 048 576	0.000 000 953 674 316 406 25
21	2 097 152	0.000 000 476 837 158 203 125
22	4 194 304	0.000 000 238 418 579 101 562 5
23	8 388 608	0.000 000 119 209 289 550 731 25
24	16 777 216	0.000 000 059 604 644 775 390 625
25	33 554 432	0.000 000 029 802 322 387 695 312 5
26	67 108 864	0.000 000 014 901 161 193 847 656 25
27	134 217 728	0.000 000 007 450 580 596 923 828 125
28	268 435 456	0.000 000 003 725 290 298 461 914 062 5
29	536 870 912	0.000 000 001 862 645 149 230 957 031 25
30	1 073 741 824	0.000 000 000 931 322 574 615 478 515 625
31	2 147 483 648	0.000 000 000 465 661 287 307 739 257 812 5
32	4 294 967 296	0.000 000 000 232 830 643 653 869 628 906 25
33	8 589 934 592	0.000 000 000 116 415 321 826 934 814 453 125
34	17 179 869 184	0.000 000 000 058 207 660 913 467 407 226 562 5
35	34 359 738 368	0.000 000 000 029 103 830 456 733 703 613 281 25
36	68 719 476 736	0.000 000 000 014 551 915 228 366 851 806 640 625
37	137 438 953 472	0.000 000 000 007 275 957 614 183 425 903 320 312 5
38	274 877 906 944	0.000 000 000 003 637 978 807 091 712 951 660 156 25
39	549 755 813 888	0.000 000 000 001 818 989 403 545 856 475 830 078 125

POWERS OF EIGHT

n	8^n	8^{-n}
0	1	1.0
1	8	0.125
2	64	0.015 625
3	512	0.001 953 125
4	40 96	0.000 244 140 625
5	32 768	0.000 030 517 578 125
6	262 144	0.000 003 814 697 265 625
7	2 097 152	0.000 000 476 837 158 203 125
8	16 777 216	0.000 000 059 604 644 775 390 625
9	134 217 728	0.000 000 007 450 580 596 923 828 125
10	1 073 741 824	0.000 000 000 931 322 574 615 478 515 625
11	8 589 934 592	0.000 000 000 116 415 321 826 934 814 453 125
12	68 719 476 736	0.000 000 000 014 551 915 228 366 851 806 640 625
13	549 755 813 888	0.000 000 000 001 818 989 403 545 856 475 830 078 125

POWERS OF SIXTEEN

n	16^n	16^{-n}
0	1	1.0
1	16	0.062 5
2	256	0.003 906 25
3	4 096	0.000 244 140 625
4	65 536	0.000 015 258 789 062 5
5	1 048 576	0.000 000 953 674 316 406 25
6	16 777 216	0.000 000 059 604 644 775 390 625
7	268 435 456	0.000 000 003 725 290 298 461 914 062 5
8	4 294 967 296	0.000 000 000 232 830 643 653 869 628 906 25
9	68 719 476 736	0.000 000 000 014 551 915 228 366 851 806 640 625

POWERS OF THREE

n	3^n	3^{-n}
0	1	1.0
1	3	0.333
2	9	0.111 11
3	27	0.037 037
4	81	0.012 345
5	243	0.004 115 2*
6	729	0.001 371 7*
7	2 187	0.000 457 24*
8	6 561	0.000 152 42*
9	19 683	0.000 050 806*
10	59 049	0.000 016 935*
11	177 147	0.000 005 644 9*
12	531 441	0.000 001 881 6*
13	1 594 323	0.000 000 627 21*
14	4 782 969	0.000 000 209 07*
15	14 348 907	0.000 000 069 695*
16	43 046 721	0.000 000 023 232*
17	129 140 163	0.000 000 007 743 8*
18	387 420 489	0.000 000 002 581 3*
19	1 162 261 467	0.000 000 000 860 44*
20	3 486 784 401	0.000 000 000 286 81*

*Note: The last figure may be in doubt and is for rounding-off purposes only.

THREE AND FOUR DIGIT BINARY EQUIVALENTS

number	three digit binary	four digit binary
0	000	0000
1	001	0001
2	010	0010
3	011	0011
4	100	0100
5	101	0101
6	110	0110
7	111	0111
8		1000
9		1001
10		1010
11		1011
12		1100
13		1101
14		1110
15		1111

TABLE 17.6. PRIME FACTORS OF NATURAL NUMBERS

n	factors	n	factors	n	factors	n	factors	n	factors
1*		51	3.17	**101**		**151**		201	3.67
2		52	$2^2.13$	102	2.3.17	152	$2^3.19$	202	2.101
3		**53**		**103**		153	$3^2.17$	203	7.29
4	2^2	54	2.3^3	104	$2^3.13$	154	2.7.11	204	$2^2.3.17$
5		55	5.11	105	3.5.7	155	5.31	205	5.41
6	2.3	56	$2^3.7$	106	2.53	156	$2^2.3.13$	206	2.103
7		57	3.19	**107**		**157**		207	$3^2.23$
8	2^3	58	2.29	108	$2^2.3^3$	158	2.79	208	$2^4.13$
9	3^2	**59**		**109**		159	3.53	209	11.19
10	2.5	60	$2^2.3.5$	110	2.5.11	160	$2^5.5$	210	2.3.5.7
11		**61**		111	3.37	161	7.23	**211**	
12	$2^2.3$	62	2.31	112	$2^4.7$	162	2.3^4	212	$2^2.53$
13		63	$3^2.7$	**113**		**163**		213	3.71
14	2.7	64	2^6	114	2.3.19	164	$2^2.41$	214	2.107
15	3.5	65	5.13	115	5.23	165	3.5.11	215	5.43
16	2^4	66	2.3.11	116	$2^2.29$	166	2.83	216	$2^3.3^3$
17		**67**		117	$3^2.13$	**167**		217	7.31
18	2.3^2	68	$2^2.17$	118	2.59	168	$2^3.3.7$	218	2.109
19		69	3.23	119	7.17	169	13^2	219	3.73
20	$2^2.5$	70	2.5.7	120	$2^3.3.5$	170	2.5.17	220	$2^2.5.11$
21	3.7	**71**		121	11^2	171	$3^2.19$	221	13.17
22	2.11	72	$2^3.3^2$	122	2.61	172	$2^2.43$	222	2.3.37
23		**73**		123	3.41	**173**		**223**	
24	$2^3.3$	74	2.37	124	2^2 31	174	2.3.29	224	$2^5.7$
25	5^2	75	3.5^2	125	5^3	175	$5^2.7$	225	$3^2.5^2$
26	2.13	76	$2^2.19$	126	$2.3^2.7$	176	$2^4.11$	226	2.113
27	3^3	77	7.11	**127**		177	3.59	**227**	
28	$2^2.7$	78	2.3.13	128	2^7	178	2.89	228	$2^2.3.19$
29		**79**		129	3.43	**179**		**229**	
30	2.3.5	80	$2^4.5$	130	2.5.13	180	$2^2.3^2.5$	230	2.5.23
31		81	3^4	**131**		**181**		231	3.7.11
32	2^5	82	2.41	132	$2^2.3.11$	182	2.7.13	232	$2^3.29$
33	3.11	**83**		133	7.19	183	3.61	**233**	
34	2.17	84	$2^2.3.7$	134	2.67	184	$2^3.23$	234	$2.3^2.13$
35	5.7	85	5.17	135	$3^3.5$	185	5.37	235	5.47
36	$2^2.3^2$	86	2.43	136	$2^3.17$	186	2.3.31	236	$2^2.59$
37		87	3.29	**137**		187	11.17	237	3.79
38	2.19	88	$2^3.11$	138	2.3.23	188	$2^2.47$	238	2.7.17
39	3.13	**89**		**139**		189	$3^3.7$	**239**	
40	$2^3.5$	90	$2.3^2.5$	140	$2^2.5.7$	190	2.5.19	240	$2^4.3.5$
41		91	7.13	141	3.47	**191**		**241**	
42	2.3.7	92	$2^2.23$	142	2.71	192	$2^6.3$	242	2.11^2
43		93	3.31	143	11.13	**193**		243	3^5
44	$2^2.11$	94	2.47	144	$2^4.3^2$	194	2.97	244	$2^2.61$
45	$3^2.5$	95	5.19	145	5.29	195	3.5.13	245	5.7^2
46	2.23	96	$2^5.3$	146	2.73	196	$2^2.7^2$	246	2.3.41
47		**97**		147	3.7^2	**197**		247	13.19
48	$2^4.3$	98	2.7^2	148	$2^2.37$	198	$2.3^2.11$	248	$2^3.31$
49	7^2	99	$3^2.11$	**149**		**199**		249	3.83
50	2.5^2	100	$2^2.5^2$	150	$2.3.5^2$	200	$2^3.5^2$	250	2.5^3

* Prime numbers are in bold face.

TABLE 17.6. (*continued*). PRIME FACTORS OF NATURAL NUMBERS

n	factors	n	factors	n	factors	n	factors	n	factors
251		301	7.43	351	$3^3.13$	**401**		451	11.41
252	$2^2.3^2.7$	302	2.151	352	$2^5.11$	402	$2.3.67$	452	$2^2.113$
253	11.23	303	3.101	**353**		403	13.31	453	3.151
254	2.127	304	$2^4.19$	354	$2.3.59$	404	$2^2.101$	454	2.227
255	$3.5.17$	305	5.61	355	5.71	405	$3^4.5$	455	$5.7.13$
256	2^8	306	$2.3^2.17$	356	$2^2.89$	406	$2.7.29$	456	$2^3.3.19$
257		**307**		357	$3.7.17$	407	11.37	**457**	
258	$2.3.43$	308	$2^2.7.11$	358	2.179	408	$2^3.3.17$	458	2.229
259	7.37	309	3.103	**359**		**409**		459	$3^3.17$
260	$2^2.5.13$	310	$2.5.31$	360	$2^3.3^2.5$	410	$2.5.41$	460	$2^2.5.23$
261	$3^2.29$	**311**		361	19^2	411	3.137	**461**	
262	2.131	312	$2^3.3.13$	362	2.181	412	$2^2.103$	462	$2.3.7.11$
263		**313**		363	3.11^2	413	7.59	**463**	
264	$2^3.3.11$	314	2.157	364	$2^2.7.13$	414	$2.3^2.23$	464	$2^4.29$
265	5.53	315	$3^2.5.7$	365	5.73	415	5.83	465	$3.5.31$
266	$2.7.19$	316	$2^2.79$	366	$2.3.61$	416	$2^5.13$	466	2.233
267	3.89	**317**		**367**		417	3.139	**467**	
268	$2^2.67$	318	$2.3.53$	368	$2^4.23$	418	$2.11.19$	468	$2^2.3^2.13$
269		319	11.29	369	$3^2.41$	**419**		469	7.67
270	$2.3^3.5$	320	$2^6.5$	370	$2.5.37$	420	$2^2.3.5.7$	470	$2.5.47$
271		321	3.107	371	7.53	**421**		471	3.157
272	$2^4.17$	322	$2.7.23$	372	$2^2.3.31$	422	2.211	472	$2^3.59$
273	$3.7.13$	323	17.19	**373**		423	$3^2.47$	473	11.43
274	2.137	324	$2^2.3^4$	374	$2.11.17$	424	$2^3.53$	474	$2.3.79$
275	$5^2.11$	325	$5^2.13$	375	3.5^3	425	$5^2.17$	475	$5^2.19$
276	$2^2.3.23$	326	2.163	376	$2^3.47$	426	$2.3.71$	476	$2^2.7.17$
277		327	3.109	377	13.29	427	7.61	477	$3^2.53$
278	2.139	328	$2^3.41$	378	$2.3^3.7$	428	$2^2.107$	478	2.239
279	$3^2.31$	329	7.47	**379**		429	$3.11.13$	**479**	
280	$2^3.5.7$	330	$2.3.5.11$	380	$2^2.5.19$	430	$2.5.43$	480	$2^5.3.5$
281		**331**		381	3.127	**431**		481	13.37
282	$2.3.47$	332	$2^2.83$	382	2.191	432	$2^4.3^3$	482	2.241
283		333	$3^2.37$	**383**		**433**		483	$3.7.23$
284	$2^2.71$	334	2.167	384	$2^7.3$	434	$2.7.31$	484	$2^2.11^2$
285	$3.5.19$	335	5.67	385	$5.7.11$	435	$3.5.29$	485	5.97
286	$2.11.13$	336	$2^4.3.7$	386	2.193	436	$2^2.109$	486	2.3^5
287	7.41	**337**		387	$3^2.43$	437	19.23	**487**	
288	$2^5.3^2$	338	2.13^2	388	$2^2.97$	438	$2.3.73$	488	$2^3.61$
289	17^2	339	3.113	**389**		**439**		489	3.163
290	$2.5.29$	340	$2^2.5.17$	390	$2.3.5.13$	440	$2^3.5.11$	490	$2.5.7^2$
291	3.97	341	11.31	391	17.23	441	$3^2.7^2$	**491**	
292	$2^2.73$	342	$2.3^2.19$	392	$2^3.7^2$	442	$2.13.17$	492	$2^2.3.41$
293		343	7^3	393	3.131	**443**		493	17.29
294	$2.3.7^2$	344	$2^3.43$	394	2.197	444	$2^2.3.37$	494	$2.13.19$
295	5.59	345	$3.5.23$	395	5.79	445	5.89	495	$3^2.5.11$
296	$2^3.37$	346	2.173	396	$2^2.3^2.11$	446	2.223	496	$2^4.31$
297	$3^3.11$	**347**		**397**		447	3.149	497	7.71
298	2.149	348	$2^2.3.29$	398	2.199	448	$2^6.7$	498	$2.3.83$
299	13.23	**349**		399	$3.7.19$	**449**		**499**	
300	$2^2.3.5^2$	350	$2.5^2.7$	400	$2^4.5^2$	450	$2.3^2.5^2$	500	$2^2.5^3$

TABLE 17.6. *(continued)* PRIME FACTORS OF NATURAL NUMBERS

n	factors	n	factors	n	factors	n	factors	n	factors
501	3.167	551	19.29	**601**		651	$3.7.31$	**701**	
502	2.251	552	$2^3.3.23$	602	$2.7.43$	652	$2^2.163$	702	$2.3^3.13$
503		553	7.79	603	$3^2.67$	**653**		703	19.37
504	$2^3.3^2.7$	554	2.277	604	$2^2.151$	654	$2.3.109$	704	$2^6.11$
505	5.101	555	$3.5.37$	605	5.11^2	655	5.131	705	$3.5.47$
506	$2.11.23$	556	$2^2.139$	606	$2.3.101$	656	$2^4.41$	706	2.353
507	3.13^2	**557**		**607**		657	$3^2.73$	707	7.101
508	$2^2.127$	558	$2.3^2.31$	608	$2^5.19$	658	$2.7.47$	708	$2^2.3.59$
509		559	13.43	609	$3.7.29$	**659**		**709**	
510	$2.3.5.17$	560	$2^4.5.7$	610	$2.5.61$	660	$2^2.3.5.11$	710	$2.5.71$
511	7.73	561	$3.11.17$	611	13.47	**661**		711	$3^2.79$
512	2^9	562	2.281	612	$2^2.3^2.17$	662	2.331	712	$2^3.89$
513	$3^3.19$	**563**		**613**		663	$3.13.17$	713	23.31
514	2.257	564	$2^2.3.47$	614	2.307	664	$2^3.83$	714	$2.3.7.17$
515	5.103	565	5.113	615	$3.5.41$	665	$5.7.19$	715	$5.11.13$
516	$2^2.3.43$	566	2.283	616	$2^3.7.11$	666	$2.3^2.37$	716	$2^2.179$
517	11.47	567	$3^4.7$	**617**		667	23.29	717	3.239
518	$2.7.37$	568	$2^3.71$	618	$2.3.103$	668	$2^2.167$	718	2.359
519	3.173	**569**		**619**		669	3.223	**719**	
520	$2^3.5.13$	570	$2.3.5.19$	620	$2^2.5.31$	670	$2.5.67$	720	$2^4.3^2.5$
521		**571**		621	$3^3.23$	671	11.61	721	7.103
522	$2.3^2.29$	572	$2^2.11.13$	622	2.311	672	$2^5.3.7$	722	2.19^2
523		573	3.191	623	7.89	**673**		723	3.241
524	$2^2.131$	574	$2.7.41$	624	$2^4.3.13$	674	2.337	724	$2^2.181$
525	$3.5^2.7$	575	$5^2.23$	625	5^4	675	$3^3.5^2$	725	$5^2.29$
526	2.263	576	$2^6.3^2$	626	2.313	676	$2^2.13^2$	726	$2.3.11^2$
527	17.31	**577**		627	$3.11.19$	**677**		**727**	
528	$2^4.3.11$	578	2.17^2	628	$2^2.157$	678	$2.3.113$	728	$2^3.7.13$
529	23^2	579	3.193	629	17.37	679	7.97	729	3^6
530	$2.5.53$	580	$2^2.5.29$	630	$2.3^2.5.7$	680	$2^3.5.17$	730	$2.5.73$
531	$3^2.59$	581	7.83	**631**		681	3.227	731	17.43
532	$2^2.7.19$	582	$2.3.97$	632	$2^3.79$	682	$2.11.31$	732	$2^2.3.61$
533	13.41	583	11.53	633	3.211	**683**		**733**	
534	$2.3.89$	584	$2^3.73$	634	2.317	684	$2^2.3^2.19$	734	2.367
535	5.107	585	$3^2.5.13$	635	5.127	685	5.137	735	$3.5.7^2$
536	$2^3.67$	586	2.293	636	$2^2.3.53$	686	2.7^3	736	$2^5.23$
537	3.179	**587**		637	$7^2.13$	687	3.229	737	11.67
538	2.269	588	$2^2.3.7^2$	638	$2.11.29$	688	$2^4.43$	738	$2.3^2.41$
539	$7^2.11$	589	19.31	639	$3^2.71$	689	13.53	**739**	
540	$2^2.3^3.5$	590	$2.5.59$	640	$2^7.5$	690	$2.3.5.23$	740	$2^2.5.37$
541		591	3.197	**641**		**691**		741	$3.13.19$
542	2.271	592	$2^4.37$	642	$2.3.107$	692	$2^2.173$	742	$2.7.53$
543	3.181	**593**		**643**		693	$3^2.7.11$	**743**	
544	$2^5.17$	594	$2.3^3.11$	644	$2^2.7.23$	694	2.347	744	$2^3.3.31$
545	5.109	595	$5.7.17$	645	$3.5.43$	695	5.139	745	5.149
546	$2.3.7.13$	596	$2^2.149$	646	$2.17.19$	696	$2^3.3.29$	746	2.373
547		597	3.199	**647**		697	17.41	747	$3^2.83$
548	$2^2.137$	598	$2.13.23$	648	$2^3.3^4$	698	2.349	748	$2^2.11.17$
549	$3^2.61$	**599**		649	11.59	699	3.233	749	7.107
550	$2.5^2.11$	600	$2^3.3.5^2$	650	$2.5^2.13$	700	$2^2.5^2.7$	750	$2.3.5^3$

TABLE 17.6. (*continued.*) PRIME FACTORS OF NATURAL NUMBERS

n	factors	n	factors	n	factors	n	factors	n	factors
751		801	$3^2.89$	851	23.37	901	17.53	951	3.317
752	$2^4.47$	802	2.401	852	$2^2.3.71$	902	$2.11.41$	952	$2^3.7.17$
753	3.251	803	11.73	**853**		903	$3.7.43$	**953**	
754	$2.13.29$	804	$2^2.3.67$	854	$2.7.61$	904	$2^3.113$	954	$2.3^2.53$
755	5.151	805	$5.7.23$	855	$3^2.5.19$	905	5.181	955	5.191
756	$2^2.3^3.7$	806	$2.13.31$	856	$2^3.107$	906	$2.3.151$	956	$2^2.239$
757		807	3.269	**857**		**907**		957	$3.11.29$
758	2.379	808	$2^3.101$	858	$2.3.11.13$	908	$2^2.227$	958	2.479
759	$3.11.23$	**809**		**859**		909	$3^2.101$	959	7.137
760	$2^3.5.19$	810	$2.3^4.5$	860	$2^2.5.43$	910	$2.5.7.13$	960	$2^6.3.5$
761		**811**		861	$3.7.41$	**911**		961	31^2
762	$2.3.127$	812	$2^2.7.29$	862	2.431	912	$2^4.3.19$	962	$2.13.37$
763	7.109	813	3.271	**863**		913	11.83	963	$3^2.107$
764	$2^2.191$	814	$2.11.37$	864	$2^5.3^3$	914	2.457	964	$2^2.241$
765	$3^2.5.17$	815	5.163	865	5.173	915	$3.5.61$	965	5.193
766	2.383	816	$2^4.3.17$	866	2.433	916	$2^2.229$	966	$2.3.7.23$
767	13.59	817	19.43	867	3.17^2	917	7.131	**967**	
768	$2^8.3$	818	2.409	868	$2^2.7.31$	918	$2.3^3.17$	968	$2^3.11^2$
769		819	$3^2.7.13$	869	11.79	**919**		969	$3.17.19$
770	$2.5.7.11$	820	$2^2.5.41$	870	$2.3.5.29$	920	$2^3.5.23$	970	$2.5.97$
771	3.257	**821**		871	13.67	921	3.307	**971**	
772	$2^2.193$	822	$2.3.137$	872	$2^3.109$	922	2.461	972	$2^2.3^5$
773		**823**		873	$3^2.97$	923	13.71	973	7.139
774	$2.3^2.43$	824	$2^3.103$	874	$2.19.23$	924	$2^2.3.7.11$	974	2.487
775	$5^2.31$	825	$3.5^2.11$	875	$5^3.7$	925	$5^2.37$	975	$3.5^2.13$
776	$2^3.97$	826	$2.7.59$	876	$2^2.3.73$	926	2.463	976	$2^4.61$
777	$3.7.37$	**827**		**877**		927	$3^2.103$	**977**	
778	2.389	828	$2^2.3^2.23$	878	2.439	928	$2^5.29$	978	$2.3.163$
779	19.41	**829**		879	3.293	**929**		979	11.89
780	$2^2.3.5.13$	830	$2.5.83$	880	$2^4.5.11$	930	$2.3.5.31$	980	$2^2.5.7^2$
781	11.71	831	3.277	**881**		931	$7^2.19$	981	$3^2.109$
782	$2.17.23$	832	$2^6.13$	882	$2.3^2.7^2$	932	$2^2.233$	982	2.491
783	$3^3.29$	833	$7^2.17$	**883**		933	3.311	**983**	
784	$2^4.7^2$	834	$2.3.139$	884	$2^2.13.17$	934	2.467	984	$2^3.3.41$
785	5.157	835	5.167	885	$3.5.59$	935	$5.11.17$	985	5.197
786	$2.3.131$	836	$2^2.11.19$	886	2.443	936	$2^3.3^2.13$	986	$2.17.29$
787		837	$3^3.31$	**887**		**937**		987	$3.7.47$
788	$2^2.197$	838	2.419	888	$2^3.3.37$	938	$2.7.67$	988	$2^2.13.19$
789	3.263	**839**		889	7.127	939	3.313	989	23.43
790	$2.5.79$	840	$2^3.3.5.7$	890	$2.5.89$	940	$2^2.5.47$	990	$2.3^2.5.11$
791	7.113	841	29^2	891	$3^4.11$	**941**		**991**	
792	$2^3.3^2.11$	842	2.421	892	$2^2.223$	942	$2.3.157$	992	$2^5.31$
793	13.61	843	3.281	893	19.47	943	23.41	993	3.331
794	2.397	844	$2^2.211$	894	$2.3.149$	944	$2^4.59$	994	$2.7.71$
795	$3.5.53$	845	5.13^2	895	5.179	945	$3^3.5.7$	995	5.199
796	$2^2.199$	846	$2.3^2.47$	896	$2^7.7$	946	$2.11.43$	996	$2^2.3.83$
797		847	7.11^2	897	$3.13.23$	**947**		**997**	
798	$2.3.7.19$	848	$2^4.53$	898	2.449	948	$2^2.3.79$	998	2.499
799	17.47	849	3.283	899	29.31	949	13.73	999	$3^3.37$
800	$2^5.5^2$	850	$2.5^2.17$	900	$2^2.3^2.5^2$	950	$2.5^2.19$	1000	$2^3.5^3$

TABLE 17.7. NATURAL SINES (COSINES) AND TANGENTS

To obtain cosine, use the formula $\cos x^\circ = \sin (90-x)^\circ$.

tangent	degree	sine, minutes 0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	proportional parts 1′	2′	3′
.00000	0	.00000	.00175	.00349	.00524	.00698	.00873	.01047	.01222	.01396	.01571	29	58	87
.01746	1	.01745	.01920	.02094	.02269	.02443	.02618	.02792	.02967	.03141	.03316	29	58	87
.03492	2	.03490	.03664	.03839	.04013	.04188	.04362	.04536	.04711	.04885	.05059	29	58	87
.05241	3	.05234	.05408	.05582	.05756	.05931	.06105	.06279	.06453	.06627	.06802	29	58	87
.06993	4	.06976	.07150	.07324	.07498	.07672	.07846	.08020	.08194	.08368	.08542	29	58	87
.08749	5	.08716	.08889	.09063	.09237	.09411	.09585	.09758	.09932	.10106	.10279	29	58	87
.10510	6	.10453	.10626	.10800	.10973	.11147	.11320	.11494	.11667	.11840	.12014	29	58	87
.12278	7	.12187	.12360	.12533	.12706	.12880	.13053	.13226	.13399	.13572	.13744	29	58	86
.14054	8	.13917	.14090	.14263	.14436	.14608	.14781	.14954	.15126	.15299	.15471	29	58	86
.15838	9	.15643	.15816	.15988	.16160	.16333	.16505	.16677	.16849	.17021	.17193	29	57	86
.17633	10	.17365	.17537	.17708	.17880	.18052	.18224	.18395	.18567	.18738	.18910	29	57	86
.19438	11	.19081	.19252	.19423	.19595	.19766	.19937	.20108	.20279	.20450	.20620	28	57	86
.21256	12	.20791	.20962	.21132	.21303	.21474	.21644	.21814	.21985	.22155	.22325	28	57	85
.23087	13	.22495	.22665	.22835	.23005	.23175	.23345	.23514	.23684	.23853	.24023	28	57	85
.24933	14	.24192	.24362	.24531	.24700	.24869	.25038	.25207	.25376	.25545	.25713	28	56	85
.26795	15	.25882	.26050	.26219	.26387	.26556	.26724	.26892	.27060	.27228	.27396	28	56	84
.28675	16	.27564	.27731	.27899	.28067	.28234	.28402	.28569	.28736	.28903	.29070	28	56	84
.30573	17	.29237	.29404	.29571	.29737	.29904	.30071	.30237	.30403	.30570	.30736	28	56	83
.32492	18	.30902	.31068	.31233	.31399	.31565	.31730	.31896	.32061	.32227	.32392	28	55	83
.34433	18	.32557	.32722	.32887	.33051	.33216	.33381	.33545	.33710	.33874	.34038	27	55	82
.36397	20	.34202	.34366	.34530	.34694	.34857	.35021	.35184	.35347	.35511	.35674	27	55	82
.38386	21	.35837	.36000	.36162	.36325	.36488	.36650	.36812	.36975	.37137	.37299	27	54	81
.40403	22	.37461	.37622	.37784	.37946	.38107	.38268	.38430	.38591	.38752	.38912	27	54	81
.42447	23	.39073	.39234	.39394	.39555	.39715	.39875	.40035	.40195	.40355	.40514	27	53	80
.44523	24	.40674	.40833	.40992	.41151	.41310	.41469	.41628	.41787	.41945	.42104	26	53	79
.46631	25	.42262	.42420	.42578	.42736	.42894	.43051	.43209	.43366	.43523	.43680	26	53	79
.48773	26	.43837	.43994	.44151	.44307	.44464	.44620	.44776	.44932	.45088	.45243	26	52	78
.50953	27	.45899	.45554	.45710	.45865	.46020	.46175	.46330	.46484	.46639	.46793	26	52	77
.53171	28	.46947	.47101	.47255	.47409	.47562	.47716	.47869	.48022	.48175	.48328	26	51	77
.55431	29	.48481	.48634	.48786	.48938	.49090	.49242	.49394	.49546	.49697	.49849	25	51	76

Tangents are recorded at intervals of one degree while sines at intervals of six minutes. The values of sines for other values of the argument can be obtained by interpolation using the columns for proportional parts. Thus $\sin 14°7' = \sin 14°6' +$ proportional parts for $1' = .24362 + .00028 = .24390$, $\sin 14°\,10' = \sin 14°\,12' -$ proportional part for $2' = .24531 - .00056 = .24475$.

TABLE 17.7 (*continued*). NATURAL SINES (COSINES) AND TANGENTS

To obtain cosine use the formula $\cos x^\circ = \sin (90-x)^\circ$

tangent	degree	sine, minutes 0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	proportional parts 1′	2′	3′
.57735	30	.50000	.50151	.50302	.50453	.50603	.50754	.50904	.51054	.51204	.51354	25	50	75
.60086	31	.51504	.51653	.51803	.51952	.52101	.52250	.52399	.52547	.52696	.52844	25	50	74
.62487	32	.52992	.53140	.53288	.53435	.53583	.53730	.53877	.54024	.54171	.54317	25	49	74
.64941	33	.54464	.54610	.54756	.54902	.55048	.55194	.55339	.55484	.55630	.55775	24	49	73
.67451	34	.55919	.56064	.56208	.56353	.56497	.56641	.56784	.56928	.57071	.57215	24	48	72
.70021	35	.57358	.57501	.57643	.57786	.57928	.58070	.58212	.58354	.58496	.58637	24	47	71
.72654	36	.58779	.58920	.59061	.59201	.59342	.59482	.59622	.59763	.59902	.60042	23	47	70
.75355	37	.60182	.60321	.60460	.60599	.60738	.60876	.61015	.61153	.61291	.61429	23	46	69
.78129	38	.61566	.61704	.61841	.61978	.62115	.62251	.62388	.62524	.62660	.62796	23	46	68
.80978	39	.62932	.63068	.63203	.63338	.63473	.63608	.63742	.63877	.64011	.64145	22	45	67
.83910	40	.64279	.64412	.64546	.64679	.64812	.64945	.65077	.65210	.65342	.65474	22	44	66
.86929	41	.65606	.65738	.65869	.66000	.66131	.66262	.66393	.66523	.66653	.66783	22	44	65
.90040	42	.66913	.67043	.67172	.67301	.67430	.67559	.67688	.67816	.67944	.68072	21	43	64
.93252	43	.68200	.68327	.68455	.68582	.68709	.68835	.68962	.69088	.69214	.69340	21	42	63
.96569	44	.69466	.69591	.69717	.69842	.69966	.70091	.70215	.70339	.70463	.70587	21	42	62
1.00000	45	.70711	.70834	.70957	.71080	.71203	.71325	.71447	.71569	.71691	.71813	20	41	61
1.03553	46	.71934	.72055	.72176	.72297	.72417	.72537	.72657	.72777	.72897	.73016	20	40	60
1.07237	47	.73135	.73254	.73373	.73491	.73610	.73728	.73846	.73963	.74080	.74198	20	39	59
1.11061	48	.74314	.74431	.74548	.74664	.74780	.74896	.75011	.75126	.75241	.75356	19	39	58
1.15037	49	.75471	.75585	.75700	.75813	.75927	.76041	.76154	.76267	.76380	.76492	19	38	57
1.19175	50	.76604	.76717	.76828	.76940	.77051	.77162	.77273	.77384	.77494	.77605	19	37	56
1.23490	51	.77715	.77824	.77934	.78043	.78152	.78261	.78369	.78478	.78586	.78694	18	36	54
1.27994	52	.78801	.78908	.79016	.79122	.79229	.79335	.79441	.79547	.79653	.79758	18	35	53
1.32704	53	.79864	.79968	.80073	.80178	.80282	.80386	.80489	.80593	.80696	.80799	17	35	52
1.37638	54	.80902	.81004	.81106	.81208	.81310	.81412	.81513	.81614	.81714	.81815	17	34	51
1.42815	55	.81915	.82015	.82115	.82214	.82314	.82413	.82511	.82610	.82708	.82806	16	33	50
1.48256	56	.82904	.83001	.83098	.83195	.83292	.83389	.83485	.83581	.83676	.83772	16	32	48
1.53986	57	.83867	.83962	.84057	.84151	.84245	.84339	.84433	.84526	.84619	.84712	16	31	47
1.60033	58	.84805	.84897	.84989	.85081	.85173	.85264	.85355	.85446	.85536	.85627	15	30	46
1.66428	59	.85717	.85806	.85896	.85985	.86074	.86163	.86251	.86340	.86427	.86515	15	30	44

Table 17.7 (*continued*). NATURAL SINES (COSINES) AND TANGENTS

To obtain cosine use the formula $\cos x^\circ = \sin (90-x)^\circ$

tangent	degree	sine: minutes 0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	proportional parts 1′	2′	3′
1.73205	**60**	.86603	.86690	.86777	.86863	.86949	.87036	.87121	.87207	.87292	.87377	14	29	43
1.80405	**61**	.87462	.87546	.87631	.87715	.87798	.87882	.87965	.88048	.88130	.88213	14	28	42
1.88073	**62**	.88295	.88377	.88458	.88539	.88620	.88701	.88782	.88862	.88942	.89021	13	27	40
1.96261	**63**	.89101	.89180	.89259	.89337	.89415	.89493	.89571	.89649	.89726	.89803	13	26	39
2.05030	**64**	.89879	.89956	.90032	.90108	.90183	.90259	.90334	.90408	.90483	.90557	13	25	38
2.14451	**65**	.90631	.90704	.90778	.90851	.90924	.90996	.91068	.91140	.91212	.91283	12	24	36
2.24604	**66**	.91355	.91425	.91496	.91566	.91636	.91706	.91775	.91845	.91914	.91982	12	23	35
2.35585	**67**	.92050	.92119	.92186	.92254	.92321	.92388	.92455	.92521	.92587	.92653	11	22	34
2.47509	**68**	.92718	.92784	.92849	.92913	.92978	.93042	.93106	.93169	.93232	.93295	11	21	32
2.60509	**69**	.93358	.93420	.93483	.93544	.93606	.93667	.93728	.93789	.93849	.93909	10	20	31
2.74748	**70**	.93969	.94029	.94088	.94147	.94206	.94264	.94322	.94380	.94438	.94495	10	19	29
2.90421	**71**	.94552	.94609	.94665	.94721	.94777	.94832	.94888	.94943	.94997	.95052	9	19	28
3.07768	**72**	.95106	.95159	.95213	.95266	.95319	.95372	.95424	.95476	.95528	.95579	9	18	26
3.27085	**73**	.95630	.95681	.95732	.95782	.95832	.95882	.95931	.95981	.96029	.96078	8	17	25
3.48741	**74**	.96126	.96174	.96222	.96269	.96316	.96363	.96410	.96456	.96502	.96547	8	16	23
3.73205	**75**	.96593	.96638	.96682	.96727	.96771	.96815	.96858	.96902	.96945	.96987	7	15	22
4.01078	**76**	.97030	.97072	.97113	.97155	.97196	.97237	.97278	.97318	.97358	.97398	7	14	20
4.33148	**77**	.97437	.97476	.97515	.97553	.97592	.97630	.97667	.97705	.97742	.97778	6	13	19
4.70463	**78**	.97815	.97851	.97887	.97922	.97958	.97992	.98027	.98061	.98096	.98129	6	12	17
5.14455	**79**	.98163	.98196	.98229	.98261	.98294	.98325	.98357	.98389	.98420	.98450	5	11	16
5.67128	**80**	.98481	.98511	.98541	.98570	.98600	.98629	.98657	.98686	.98714	.98741	5	10	14
6.31375	**81**	.98769	.98796	.98823	.98849	.98876	.98902	.98927	.98953	.98978	.99002	4	9	13
7.11537	**82**	.99027	.99051	.99075	.99098	.99122	.99144	.99167	.99189	.99211	.99233	4	8	11
8.14435	**83**	.99255	.99276	.99297	.99317	.99337	.99357	.99377	.99396	.99415	.99434	3	7	10
9.51436	**84**	.99452	.99470	.99488	.99506	.99523	.99540	.99556	.99572	.99588	.99604	3	6	8
11.43005	**85**	.99619	.99635	.99649	.99664	.99678	.99692	.99705	.99719	.99731	.99744	2	5	7
14.30067	**86**	.99756	.99768	.99780	.99792	.99803	.99813	.99824	.99834	.99844	.99854	2	4	5
19.08114	**87**	.99863	.99872	.99881	.99889	.99897	.99905	.99912	.99919	.99926	.99933	1	3	4
28.63625	**88**	.99939	.99945	.99951	.99956	.99961	.99966	.99970	.99974	.99978	.99982	1	2	2
57.28996	**89**	.99985	.99988	.99990	.99993	.99995	.99996	.99998	.99999	.99999	1.00000	0	1	1

17.8. Bernoulli and Euler Numbers

The Bernoulli numbers B_n and Euler numbers E_n of order 1, of Table 17.8 are defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

$$\left(\frac{2}{e^t+e^{-t}}\right) = \text{Sech } t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

Note that for odd values of n both B_n and E_n are equal to zero, excluding of course B_1 which is equal to $-\frac{1}{2}$. The values of the first few numbers are

n	0	1	2	4	6	8	10	12
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$
E_n	1	0	-1	5	-61	1385	-50521	270265

Computing the sum of integral powers of integers. The sum $S_p(N) = 1^p + 2^p + \ldots + N^p$ is frequently needed in statistical work. For example consider a random sample of size n drawn with replacement from a finite population of N units and let V denote the number of distinct units appearing in the sample. The expected value of $1/V$ can be expressed as $E(1/V) = S_{n-1}(N)/N^{-n}$. In terms of Bernoulli numbers, $S_p(N) = \sum_{r=0}^{p} \left[\binom{p+1}{r} B_r (N+1)^{p-r+1} \right] / (p+1)$. We have thus

p	$S_p(N)$
1	$N(N+1) \div 2$
2	$N(N+1)(2N+1) \div 6$
3	$N^2(N+1)^2 \div 4$
4	$N(N+1)(2N+1)(3N^2+3N-1) \div 30$
5	$N^2(N+1)^2(2N^2+2N-1) \div 12$
6	$N(N+1)(2N+1)(3N^4+6N^3-3N+1) \div 42$
7	$N^2(N+1)^2(3N^4+6N^3-N^2-4N+2) \div 24$
8	$N(N+1)(2N+1)(5N^6+15N^5+5N^4-15N^3-N^2+9N-3) \div 90$

TABLE 17.8. BERNOULLI AND EULER NUMBERS AND THEIR LOGARITHMS

n	$\log_{10}\lvert B_{2n}\rvert$	$\lvert B_{2n}\rvert$*	$\log_{10}\lvert E_{2n}\rvert$	$\lvert E_{2n}\rvert$
1	− .77815 12504	.16666 6667	0.0000 0000 0000	1
2	−1.47712 12547	.03333 3333	0.6989 7000 4336	5
3	−1.62324 92904	.02380 9524	1.7853 2983 5011	61
4	−1.47712 12547	.03333 3333	3.1414 4977 3400	1385
5	−1.12057 39312	.07575 7576	4.7034 7193 8284	5 0521
6	− .59668 45997	.25311 3553	6.4318 0828 6305	270 2765
7	0.06694 67896	1.1666 6667	8.2996 4016 2027	19936 0981
8	0.85077 83327	7.0921 5686	10.2876 1167 6568	1939151 2145
9	1.74013 50433	54.971 1779	12.3810 9335 1978	2404879 675(3)
10	2.72355 76597	529.12 4242	14.5686 3719 4867	3703711 882(5)
11	3.79183 95878	6192.1 2319	16.8410 3941 6358	6934887 439(7)
12	4.93741 88511	86580. 2531	19.1907 3874 0073	1551453 416(10)
13	6.15397 24516	14255 17.17	21.6114 1234 2656	4087072 509(12)
14	7.43613 45056	27298 231.1	24.0976 9438 4097	1252259 641(15)
15	8.77929 40203	60158 0874	26.6449 7388 2655	4415438 932(17)
16	10.17944 59554	15116 3158(2)	29.2492 4580 0749	1775193 916(20)
17	11.63307 90755	42961 4643(3)	31.9069 9890 3609	8072329 924(22)
18	13.13708 98839	13711 6552(5)	34.6151 2969 4666	4122206 034(25)
19	14.68871 54679	48833 2319(6)	37.3708 7526 1289	2348958 053(28)
20	16.28548 03295	19296 5793(8)	40.1717 6010 5584	1485115 072(31)
21	17.92515 37399	84169 3048(9)	43.0155 5349 8641	1036462 273(34)
22	19.60571 51352	40338 0719(11)	45.9002 3487 6646	7947579 423(36)
23	21.32532 57440	21150 7486(13)	48.8239 6546 8043	6667537 517(39)
24	23.08230 51026	12086 6265(15)	51.7850 6480 9294	6096278 646(42)
25	24.87511 14502	75008 6675(16)	54.7819 9113 9598	6053285 248(45)
26	26.70232 52332	50387 7810(18)	57.8133 2490 5271	6506162 487(48)
27	28.56263 51260	36528 7765(20)	60.8777 5478 0634	7546659 939(51)
28	30.45482 61057	28498 7693(22)	63.9740 6574 3074	9420321 896(54)
29	32.37776 92183	23865 4275(24)	67.1011 2883 8249	1262201 925(58)
30	34.33041 27436	21399 9493(26)	70.2578 9234 6215	1810891 150(61)
31	36.31177 45314	20500 9757(28)	73.4433 7411 6664	2775710 170(64)
32	38.32093 53181	20938 0059(30)	76.6566 5488 6041	4535810 333(67)
33	40.35703 28735	22752 6965(32)	79.8968 7242 4165	7886284 207(70)
34	42.41925 68522	26257 7103(34)	83.1632 1638 5512	1456184 438(74)
35	44.50684 42463	32125 0821(36)	86.4549 2376 2203	2850517 832(77)
36	46.61907 53547	41598 2782(38)	89.7712 7485 3293	5905747 208(80)
37	48.75527 01978	56920 6955(40)	93.1115 8967 9184	1292973 664(84)
38	50.91478 53168	82183 6294(42)	96.4752 2478 0696	2986928 183(87)
39	53.09701 09079	12502 9043(45)	99.8615 7035 4499	7270601 714(90)
40	55.30136 82495	20015 5832(47)	103.2700 4767 8721	1862291 576(94)
41	57.52730 73841	33674 9829(49)	106.7001 0679 5923	5013104 941(97)
42	59.77430 50258	59470 9705(51)	110.1512 2442 0201	1416525 576(101)
43	62.04186 26660	11011 9103(54)	113.6229 0204 3198	4196643 164(104)
44	64.32950 48541	21355 2595(56)	117.1146 6421 3908	1302159 591(108)
45	66.63677 76334	43328 8970(58)	120.6260 5697 5931	4227240 686(111)
46	68.96324 71164	91885 5282(60)	124.1566 4644 1537	1434321 279(115)
47	71.30849 81818	20346 8968(63)	127.7060 1748 9631	5081799 072(118)
48	73.67213 32834	47003 8340(65)	131.2737 7257 3899	1878332 936(122)
49	76.05377 13567	11318 0434(68)	134.8595 3062 9797	7236534 381(125)
50	78.45304 68146	28382 2496(70)	138.4629 2607 0334	2903528 347(129)

Note: For larger values of n, values of $\lvert B_{2n}\rvert$ and $\lvert E_{2n}\rvert$ are given correct to 9 and 10 significant digits respectively. The number in parenthesis following $\lvert B_{2n}\rvert$ and $\lvert E_{2n}\rvert$ is the power of 10 by which the given tabular quantity must be multiplied.

*B_{2n} is positive if n is odd and negative if n is even while E_{2n} is positive if n is even and negative if n is odd.

TABLE 17.9. COMMON LOGARITHMS
(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
100	00 0000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	*0300	0724	1147	1570	1993	2415	424
103	01 2837	3259	3680	4100	4521	4940	5360	5779	6197	6616	420
104	7033	7451	7868	8284	8700	9116	9532	9947	*0361	0775	416
105	02 1189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*0195	0600	1004	1408	1812	2216	2619	3021	404
108	03 3424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*0207	0602	0998	397
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
112	9218	9606	9993	*0380	0766	1153	1538	1924	2309	2694	386
113	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524	383
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*0320	379
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
117	8186	8557	8928	9298	9668	*0038	0407	0776	1145	1514	370
118	07 1882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	9181	9543	9904	*0266	0626	0987	1347	1707	2067	2426	360
121	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
22	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	354
123	9905	*0258	0611	0963	1315	1667	2018	2370	2721	3071	352
124	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*0026	346
126	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*0253	338
129	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*0245	330
132	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*0012	323
135	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219	320
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
138	9879	*0194	0508	0822	1136	1450	1763	2076	2389	2702	314
139	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	*0142	0449	0756	1063	1370	1676	1982	307
142	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*0168	0469	0769	1068	301
145	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055	298
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	296
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	294
148	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895	292
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	290

*An asterisk and a bold figure indicate that a change has occured in the first two figures of the mantissa, shown separately in the first column immediately following the number. Thus log 1414 = 3.150449.

To obtain natural logarithm (to base e) multiply by 2.3025851.

TABLE 17.9. *(continued)*. COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
150	17 6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	*0126	0413	0699	0986	1272	1558	287
152	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	*0051	281
155	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	*0029	0303	0577	0850	1124	274
159	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	*0051	0319	0586	0853	1121	1388	1654	1921	267
163	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*0193	256
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	*0050	0300	250
174	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*0176	245
178	25 0420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	*0213	0446	0679	0912	1144	1377	1609	233
187	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	8754	8982	9211	9439	9667	9895	*0123	0351	0578	0806	228
191	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	224
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	*0161	0378	0595	0813	218

*See footnote on page 175

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	214
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	*0056	0268	0481	0693	0906	1118	1330	1542	212
205	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	*0008	0211	203
214	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	0047	0246	199
219	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054	194
224	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	192
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	0025	0215	0404	0593	0783	0972	1161	1350	1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	*0143	0328	0513	0698	0883	185
235	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	*0030	181
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	*0051	0228	0405	0582	0759	177
246	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
250	39 7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	*0020	0192	0365	0538	0711	0883	1056	1228	173
252	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	*0102	0271	0440	0609	0777	0946	1114	1283	1451	169
258	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	166
263	9956	*0121	0286	0451	0616	0781	0945	1110	1275	1439	165
264	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	*0075	0236	0398	0559	0720	0881	1042	1203	161
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809	160
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	*0122	0279	0437	0594	0752	158
276	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	156
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	*0095	154
282	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	*0146	0296	0447	0597	0748	151
289	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	148
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	*0116	0263	0410	0557	9704	0851	0998	1145	147
296	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
300	47 7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	144
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	*0099	0239	0380	0520	0661	0801	0941	1081	1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	140
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	*0099	0236	0374	0511	0648	0785	0922	137
317	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	*0009	0143	0277	0411	134
324	51 0545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	134
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	*0090	0221	0353	0484	0615	0745	0876	1007	131
332	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	*0072	128
339	53 0200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	*0079	0204	125
347	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
350	54 4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	*0106	122
355	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	122
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	*0026	0146	0265	0385	0504	0624	0743	0863	0982	119
364	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	118
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	*0076	0193	0309	0426	117
372	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	9784	9898	*0012	0126	0241	0355	0469	0583	0697	0811	114
381	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	*0061	0173	0284	0396	0507	0619	0730	0842	0953	112
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	*0101	0210	0319	0428	0537	0646	0755	0864	109
399	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109

* See footnote on page 198

TABLE 17.9 (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
400	60 2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	*0021	0128	0234	0341	0447	0554	107
408	61 0660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	104
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	*0032	104
417	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	*0021	0123	0224	0326	102
427	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	101
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	100
436	9486	9586	9686	9785	9885	9984	*0084	0183	0283	0382	100
437	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	99
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	98
446	9335	9432	9530	9627	9724	9821	9919	*0016	0113	0210	97
447	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
450	65 3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	*0011	0106	0201	0296	0391	0486	0581	0676	0771	95
458	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	94
460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	*0060	0153	93
468	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	92
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	*0063	0154	0245	91
479	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151	90
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	*0019	0107	89
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993	88
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	87
501	9838	9924	*0011	0098	0184	0271	0358	0444	0531	0617	87
502	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	*0033	85
513	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	84
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	*0077	83
525	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	83
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	*0055	0136	0217	0298	0378	0459	0540	0621	0702	81
538	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	80
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	*0047	0126	0205	0284	79

See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	78
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	*0045	0123	0200	0277	0354	0431	77
563	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	76
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	9668	9743	9819	9894	9970	*0045	0121	0196	0272	0347	75
576	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	*0042	74
589	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72

* See footnote on page 196

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
600	77 8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	*0029	0101	0173	0245	72
603	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	72
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	*0004	0074	0144	0215	70
617	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	68
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	68
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
742	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	68
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	*0031	0098	0165	67
646	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
650	81 2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	66
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	9544	9610	9676	9741	9807	9873	9939	*0004	0070	0136	66
661	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	*0011	0075	0139	0204	0268	0332	0396	0460	0525	64
677	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	64
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	*0043	63
692	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	60
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	9739	9799	9859	9918	9978	*0038	0098	0158	0218	0278	60
725	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	9818	9877	9935	9994	*0053	0111	0170	0228	0287	0345	59
742	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
750	87 5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	58
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	*0013	0070	0127	0185	57
759	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	56
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	*0030	0086	0141	0197	0253	0309	0365	56
777	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	*0039	0094	0149	0203	0258	0312	55
795	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	54
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
800	90 3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	*0037	54
813	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	*0019	0071	52
832	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
850	92 9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	*0032	0083	0134	0185	0236	0287	0338	0389	51
852	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	50
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	50
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	*0024	0073	0121	0170	0219	0267	0316	49
892	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	48
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48

* See footnote on page 198

TABLE 17.9. *(continued).* COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	*0042	0090	0138	0185	0233	0280	0328	0376	0423	48
913	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	48
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	*0021	0068	0114	0161	0207	0254	0300	46
934	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46

* See footnote on page 198

TABLE 17.9. (*continued*). COMMON LOGARITHMS

(six-figure)

number	0	1	2	3	4	5	6	7	8	9	difference
950	97 7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412	46
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	44
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	*0028	0072	0117	0161	0206	0250	0294	44
978	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	43
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

* See footnote on page 198

TABLE 18.1. LIST OF SQUARES UPTO ORDER 6×6

To select a latin square at random :

Suppose a 6×6 Latin square is required. Choose a random number from 1 to 9408 the largest key number recorded under the last square. If the random number chosen is 3486, then select square number IV, since 3486 is in the range of key numbers 3241–4320 under that square. Next permute all the rows and all the columns of the selected Latin square at random and assign the letters to the treatments also at random. For obtaining a random permutation consult Table 19.1 and 19e for introductory note. The procedure is similar for Latin squares of sizes 4×4 and 5×5, using the key numbers recorded. For squares of higher dimension one could use one of the orthogonal squares given in Table 18.2 and permute its rows, columns and treatment numbers independently at random.

The 4×4 Latin Squares

I	II
A B C D	A B C D
B A D C	B A D C
C D B A	C D A B
D C A B	D C B A
1–3	4

The 5×5 Latin Squares

I	II	III
A B C D E	A B C D E	A B C D E
B A E C D	B A D E C	B C E A D
C D A E B	C E A B D	C E D B A
D E B A C	D C E A B	D A B E C
E C D B A	E D B C A	E D A C B
1–25	26–50	51–56

The 6×6 Latin Squares

I	II	III	IV	V
A B C D E F	A B C D E F	A B C D E F	A B C D E F	A B C D E F
B C F A D E	B C F E A D	B C F E A D	B A F E C D	B A E F C D
C F B E A D	C F B A D E	C F B A D E	C F B A D E	C F B A D E
D E A B F C	D A E B F C	D E A B F C	D C E B F A	D E A B F C
E A D F C B	E D A F C B	E A D F C B	E D A F B C	E D F C B A
F D E C B A	F E D C B A	F D E C B A	F E D C A B	F C D E A B
0001–1080	1081–2160	2161–3240	3241–4320	4321–5400

VI	VII	VIII	IX	X
A B C D E F	A B C D E F	A B C D E F	A B C D E F	A B C D E F
B A E C F D	B A F E D C	B A F E C D	B C D E F A	B A E F C D
C F B A D E	C E B F A D	C F B A D E	C E A F B D	C F A E D B
D E F B C A	D C A B F E	D E A B F C	D F B A C E	D C B A F E
E D A F B C	E F D C B A	E C D F B A	E D F B A C	E D F C B A
F C D E A B	F D E A C B	F D E C A B	F A E C D B	F E D B A C
5401–5940	5941–6480	6481–7020	7021–7560	7561–7920

XI	XII	XIII	XIV	XV
A B C D E F	A B C D E F	A B C D E F	A B C D E F	A B C D E F
B A F C D E	B A E F C D	B C F A D E	B C A F D E	B C A F D E
C E A B F D	C F A B D E	C F B E A D	C A B E F D	C A B E F D
D F E A C B	D E B A F C	D A E B F C	D F E B A C	D F E B C A
E C D F B A	E D F C B A	E D A F C B	E D F C B A	E D F A B C
F D B E A C	F C D E A B	F E D C B A	F E D A C B	F E D C A B
7921–8280	8281–8640	8641–8820	8821–8940	8941–9060

XVI	XVII	XVIII	XIX	XX
A B C D E F	A B C D E F	A B C D E F	A B C D E F	A B C D E F
B C A E F D	B C A F D E	B C A E F D	B A F E D C	B A D F C E
C A B F D E	C A B E F D	C A B F D E	C D A B F E	C F A E B D
D E F B A C	D F E B A C	D F E B A C	D F E A C B	D E B A F C
E F D A C B	E D F A C B	E D F C B A	E C B F A D	E D F C A B
F D E C B A	F E D C B A	F E D A C B	F E D C B A	F C E B D A
9061–9180	9181–9240	9241–9280	9281–9316	9317–9352

XXI	XXII
A B C D E F	A B C D E F
B A E C F D	B C A F D E
C E A F D B	C A B E F D
D C F A B E	D E F A B C
E F D B A C	E F D C A B
F D B E C A	F D E B C A
9353–9388	9389–9408

TABLE 18.2. SETS OF MUTUALLY ORTHOGONAL SQUARES

3 × 3

I	II
1 2 3	1 2 3
2 3 1	3 1 2
3 1 2	2 3 1

4 × 4

I	II	III
1 2 3 4	1 2 3 4	1 2 3 4
2 1 4 3	3 4 1 2	4 3 2 1
3 4 1 2	4 3 2 1	2 1 4 3
4 3 2 1	2 1 4 3	3 4 1 2

TABLE 18.2. (*continued*). SETS OF MUTUALLY ORTHOGONAL LATIN SQUARES

5 × 5

I	1	2	3	4	5
	2	3	4	5	1
	3	4	5	1	2
	4	5	1	2	3
	5	1	2	3	4

II	1	2	3	4	5
	3	4	5	1	2
	5	1	2	3	4
	2	3	4	5	1
	4	5	1	2	3

III	1	2	3	4	5
	4	5	1	2	3
	2	3	4	5	1
	5	1	2	3	4
	3	4	5	1	2

IV	1	2	3	4	5
	5	1	2	3	4
	4	5	1	2	3
	3	4	5	1	2
	2	3	4	5	1

7 × 7

I	1	2	3	4	5	6	7
	2	3	4	5	6	7	1
	3	4	5	6	7	1	2
	4	5	6	7	1	2	3
	5	6	7	1	2	3	4
	6	7	1	2	3	4	5
	7	1	2	3	4	5	6

II	1	2	3	4	5	6	7
	3	4	5	6	7	1	2
	5	6	7	1	2	3	4
	7	1	2	3	4	5	6
	2	3	4	5	6	7	1
	4	5	6	7	1	2	3
	6	7	1	2	3	4	5

III	1	2	3	4	5	6	7
	4	5	6	7	1	2	3
	7	1	2	3	4	5	6
	3	4	5	6	7	1	2
	6	7	1	2	3	4	5
	2	3	4	5	6	7	1
	5	6	7	1	2	3	4

IV	1	2	3	4	5	6	7
	5	6	7	1	2	3	4
	2	3	4	5	6	7	1
	6	7	1	2	3	4	5
	3	4	5	6	7	1	2
	7	1	2	3	4	5	6
	4	5	6	7	1	2	3

V	1	2	3	4	5	6	7
	6	7	1	2	3	4	5
	4	5	6	7	1	2	3
	2	3	4	5	6	7	1
	7	1	2	3	4	5	6
	5	6	7	1	2	3	4
	3	4	5	6	7	1	2

VI	1	2	3	4	5	6	7
	7	1	2	3	4	5	6
	6	7	1	2	3	4	5
	5	6	7	1	2	3	4
	4	5	6	7	1	2	3
	3	4	5	6	7	1	2
	2	3	4	5	6	7	1

8 × 8

I	1	2	3	4	5	6	7	8
	2	1	4	3	6	5	8	7
	3	4	1	2	7	8	5	6
	4	3	2	1	8	7	6	5
	5	6	7	8	1	2	3	4
	6	5	8	7	2	1	4	3
	7	8	5	6	3	4	1	2
	8	7	6	5	4	3	2	1

II	1	2	3	4	5	6	7	8
	5	6	7	8	1	2	3	4
	2	1	4	3	6	5	8	7
	6	5	8	7	2	1	4	3
	7	8	5	6	3	4	1	2
	3	4	1	2	7	8	5	6
	8	7	6	5	4	3	2	1
	4	3	2	1	8	7	6	5

III	1	2	3	4	5	6	7	8
	7	8	5	6	3	4	1	2
	5	6	7	8	1	2	3	4
	3	4	1	2	7	8	5	6
	8	7	6	5	4	3	2	1
	2	1	4	3	6	5	8	7
	4	3	2	1	8	7	6	5
	6	5	8	7	2	1	4	3

IV	1	2	3	4	5	6	7	8
	8	7	6	5	4	3	2	1
	7	8	5	6	3	4	1	2
	2	1	4	3	6	5	8	7
	4	3	2	1	8	7	6	5
	5	6	7	8	1	2	3	4
	6	5	8	7	2	1	4	3
	3	4	1	2	7	8	5	6

V	1	2	3	4	5	6	7	8
	4	3	2	1	8	7	6	5
	8	7	6	5	4	3	2	1
	5	6	7	8	1	2	3	4
	6	5	8	7	2	1	4	3
	7	8	5	6	3	4	1	2
	3	4	1	2	7	8	5	6
	2	1	4	3	6	5	8	7

VI	1	2	3	4	5	6	7	8
	6	5	8	7	2	1	4	3
	4	3	2	1	8	7	6	5
	7	8	5	6	3	4	1	2
	3	4	1	2	7	8	5	6
	8	7	6	5	4	3	2	1
	2	1	4	3	6	5	8	7
	5	6	7	8	1	2	3	4

VII	1	2	3	4	5	6	7	8
	3	4	1	2	7	8	5	6
	6	5	8	7	2	1	4	3
	8	7	6	5	4	3	2	1
	2	1	4	3	6	5	8	7
	4	3	2	1	8	7	6	5
	5	6	7	8	1	2	3	4
	7	8	5	6	3	4	1	2

9 × 9

I	1	2	3	4	5	6	7	8	9
	2	3	1	5	6	4	8	9	7
	3	1	2	6	4	5	9	7	8
	4	5	6	7	8	9	1	2	3
	5	6	4	8	9	7	2	3	1
	6	4	5	9	7	8	3	1	2
	7	8	9	1	2	3	4	5	6
	8	9	7	2	3	1	5	6	4
	9	7	8	3	1	2	6	4	5

II	1	2	3	4	5	6	7	8	9
	7	8	9	1	2	3	4	5	6
	4	5	6	7	8	9	1	2	3
	2	3	1	5	6	4	8	9	7
	8	9	7	2	3	1	5	6	4
	5	6	4	8	9	7	2	3	1
	3	1	2	6	4	5	9	7	8
	9	7	8	3	1	2	6	4	5
	6	4	5	9	7	8	3	1	2

III	1	2	3	4	5	6	7	8	9
	9	7	8	3	1	2	6	4	5
	5	6	4	8	9	7	2	3	1
	6	4	5	9	7	8	3	1	2
	2	3	1	5	6	4	8	9	7
	7	8	9	1	2	3	4	5	6
	8	9	7	2	3	1	5	6	4
	4	5	6	7	8	9	1	2	3
	3	1	2	6	4	5	9	7	8

VI	1	2	3	4	5	6	7	8	9
	8	9	7	2	3	1	5	6	4
	6	4	5	9	7	8	3	1	2
	9	7	8	3	1	2	6	4	5
	4	5	6	7	8	9	1	2	3
	2	3	1	5	6	4	8	9	7
	5	6	4	8	9	7	2	3	1
	3	1	2	6	4	5	9	7	8
	7	8	9	1	2	3	4	5	6

V	1	2	3	4	5	6	7	8	9
	3	1	2	6	4	5	9	7	8
	2	3	1	5	6	4	8	9	7
	7	8	9	1	2	3	4	5	6
	9	7	8	3	1	2	6	4	5
	8	9	7	2	3	1	5	6	4
	4	5	6	7	8	9	1	2	3
	6	4	5	9	7	8	3	1	2
	5	6	4	8	9	7	2	3	1

VI	1	2	3	4	5	6	7	8	9
	4	5	6	7	8	9	1	2	3
	7	8	9	1	2	3	4	5	6
	3	1	2	6	4	5	9	7	8
	6	4	5	9	7	8	3	1	2
	9	7	8	3	1	2	6	4	5
	2	3	1	5	6	4	8	9	7
	5	6	4	8	9	7	2	3	1
	8	9	7	2	3	1	5	6	4

VII	1	2	3	4	5	6	7	8	9
	5	6	4	8	9	7	2	3	1
	9	7	8	3	1	2	6	4	5
	8	9	7	2	3	1	5	6	4
	3	1	2	6	4	5	9	7	8
	4	5	6	7	8	9	1	2	3
	6	4	5	9	7	8	3	1	2
	7	8	9	1	2	3	4	5	6
	2	3	1	5	6	4	8	9	7

VIII	1	2	3	4	5	6	7	8	9
	6	4	5	9	7	8	3	1	2
	8	9	7	2	3	1	5	6	4
	5	6	4	8	9	7	2	3	1
	7	8	9	1	2	3	4	5	6
	3	1	2	6	4	5	9	7	8
	9	7	8	3	1	2	6	4	5
	2	3	1	5	6	4	8	9	7
	4	5	6	7	8	9	1	2	3

10 × 10

I	0	1	2	3	4	5	6	7	8	9
	1	2	0	6	7	3	9	3	4	5
	2	0	1	5	6	7	8	9	3	4
	3	7	8	0	1	4	2	5	9	6
	4	8	9	7	0	1	5	2	6	3
	5	9	3	4	8	0	1	6	2	7
	6	3	4	8	5	9	0	1	7	2
	7	4	5	2	9	6	3	0	1	8
	8	5	6	9	2	3	7	4	0	1
	9	6	7	1	3	2	4	8	5	0

II	0	1	2	3	4	5	6	7	8	9
	2	0	1	8	9	3	4	5	6	7
	1	2	0	4	5	6	7	8	9	3
	7	3	9	6	8	0	5	2	1	4
	8	4	3	5	7	9	0	6	2	1
	9	5	4	1	6	8	3	0	7	2
	3	6	5	2	1	7	9	4	0	8
	4	7	6	9	2	1	8	3	5	0
	5	8	7	0	3	2	1	9	4	6
	6	9	8	7	0	4	2	1	3	5

a. Description of the table

Each row of digits in Table 19.1 contains a serial number of row, and a random permutation of numers 0, 1, ..., 9 followed by 40 random digits in 40 columns arranged in sets of 4. The serial numbers of the columns of random digits are indicated in the bottom line of each page so that each random digit can be identified by a row number and a column number. There are altogether 5,000 four digited random numbers (equivalent to 10,000 two digited or 20,000 one digited random numbers). They have been compiled from a number of existing random number tables. The random numbers so compiled have been examined through standard tests of randomness. No serious lack of randomness was revealed.

In using Table 19.1 we need a starting point identified by a row and a column. There are no set rules for the choice of a starting point except that no preference is shown to particular page, row or column and the choice is made without prior inspection of the numbers themselves. Some random mechanism may be adopted for locating the starting point, specially when the random number table is repeatedly used for the selection of numbers (see sub-section **f** of this Chapter in this connection).

Some of the uses of Table 19.1 are given below.

b. Simple random sampling from a list

(i) *A straightforward method.* Suppose we have to sample 5 households from a list of 23, serially numbered 0, 1, ..., 22.

Locate a starting point of random digits and consider two adjacent columns. Read two digited numbers either upwards or downwards or diagonally and record the first five numbers that lie in 0–22. If sampling is without replacement continue reading till five distinct numbers are obtained. Suppose we start from row 135 and read downwards the two digited numbers in columns 3 and 4; the selected households are 20, 3, 1, 20, 3 if repetition is allowed and 20, 3, 1, 12, 18 without repetition.

(ii) *The method of inflated range.* In the above method we have to reject all numbers greater than 22, which on an average amounts to 77% of the numbers read. To reduce the number of rejections, consider the range of numbers from 0 to $23k-1$ where k is chosen such that $23k$ is nearest to, but does not exceed, a power of 10. In the present example $k = 4$ gives the range 0 to 91. Choosing two columns as before select the first five two digited numbers in the rauge 0–91. Each number chosen is replaced by the remainder after dividing by 23 to obtain a number in the range 0–22. Thus, using the same starting point as in (i) above the numbers are 80, 62, 63, 25, 53 which give the sample 11, 16, 17, 2, 7.

Alternatively when k is small as in the present example the number chosen could be divided by k and the quotient taken as the number finally selected. Thus in the example considered above, the numbers 80, 62, 63, 25 and 53 on division by $k = 4$, lead to the sample 20, 15, 15, 6 and 13.

(iii) *Independent choice of the first digit.* The method of inflated range reduces the rejection of random numbers at the expense of a tedious operation of repeated division by a given number. An alternative method due to Matthai is as follows.

To select five numbers at random from 0 to 383, locate a starting point and record two digited numbers (one less than the number of digits in the given number). To each of these numbers prefix a digit at random from 0 to 3. This could be done, for example, by considering the first number from among 0 to 3 in the random permutation that appears in the same row. A three digited number, so obtained, is rejected if it exceeds 383. Thus with the columns 9 and 10 from row 271 as the starting point and reading downwords the numbers selected are as follows: $\underline{0}53$, $\underline{2}95$, $\underline{0}00$, $\underline{1}95$, $\underline{3}34$ where in, the digits underlined are prefixed as indicated.

This method is also useful when for example one has to select numbers in the range 3845–8962. Here one selects a three digited number at random to which is prefixed a digit chosen in the range 3–8. The random permutation in the row could be used to select a random number in the range 3–8. The number finally obtained is accepted if it falls in the range 3845–8962. Otherwise it is rejected and another number is drawn in the same way.

c. Sampling with probabilities proportional to size (pps)

(i) *The method of cumulated totals.* Select five villages from a list of 23 with probabilities proportional to size of the village

serial no of village	size	cumulated totals (c.t.)
1	19	19
2	207	226
3	72	298
.	.	.
.	.	.
.	.	.
22	28	883
23	120	1003

Select five random numbers from 1 to 1003 (the last c.t.). If a chosen number is greater than the c.t. for village i and less than or equal to the c.t. for village $(i+1)$, then the village selected is $(i+1)$. Thus if the first random number chosen is 227, the village selected is 3. Similarly the villages corresponding to the second and subsequent random numbers are determined.

(ii) *A two stage selection method.* This is useful particularly when the sizes are not numerically specified nor is it intended to determine all of them beforehand, for example, in selecting crop plots with probability proportional to area etc. The method, however, requires the prior knowledge of a number S which equals or exceeds the largest of the sizes. Let 210 be that number in the above example. The procedure due to Hajek and Lahiri is as follows.

Select a number x at random from 1 to 23 and another number y from 1 to $S = 210$. If the size of village x is $\leqslant y$ then village x is chosen; otherwise, the pair of selected numbers (x, y) is rejected and another pair is considered. If a sample of 5 is required the above procedure is continued till 5 villages get selected. This method involves rejection of a large number of selected pairs if the sizes of the villages are very disproportionate. In such cases a large village may have to be split into smaller units with smaller sizes (adding upto the size of the village). Each such unit is given a separate serial number. The original village is selected if any one of its constituent units gets selected in the process.

(iii) *Cluster sampling.* Draw a cluster of four villages with probability proportional to sum of the sizes.

One method is to list all the $\binom{23}{4} = 8855$ possible clusters and their sizes. The size of any cluster is equal to the sum of the sizes of the four villages in it. Now choose a cluster with probability proportional to size by the method described in (i) or (ii) of **19c**. A simpler technique is, however, to draw one village from 1 to 23 with probability proportional to size and three villages at random with equal probability and without replacement from the remaining 22.

(iv) *Simple random sampling from separate lists.* Select a household from six streets containing 17, 32, 28, 47, 56 and 12 houses respectively.

One method is to make a serial listing of all the 192 households and select the required number in the usual way. An alternative method is to select a number from 1 to 6 specifying a street, and another number from 1 to 56 (56 being the maximum number of households in a street) specifying a household on the street. If there is no household corresponding to the second number in the selected street the pair of selected numbers is rejected and another pair is considered.

d. Model sampling

(i) *Uniform distribution over the interval* (0, 1) : $R(0, 1)$. To draw a random observation from the uniform distribution over (0, 1), start with a decimal point and record the digits in the sequence read from the random number table. The number of digits to be retained is determined by the accuracy needed in the observation. Thus selecting the 30th row and 4th column as the starting point and reading the digits horizontally, the observation is 0.04100526. The observation correct to 4 places is 0.0410.

(ii) *Discrete distribution.* This is a special case of sampling with varying probabilities (see **19c**) where the number of elements may be finite or infinite. Let the discrete variable X take the values 0, 1, 2, ... with probabilities $p_0, p_1, p_2, \ldots$. First draw an observation u from the uniform distribution $R(0, 1)$ as indicated in (i) above. Then determine x such that

$$p_0+p_1+\ldots p_{x-1} < u \leqslant p_0+p_1+\ldots+p_x.$$

The number x constitutes a random observation on X.

(iii) *Continuous distributions with cumulative distribution function (cdf), $F(x)$.* Let u be a random observation from the uniform distribution $R(0, 1)$. The value of x for which $F(x) = u$ provides a random observation from the continuous distribution with cdf $F(x)$. In the absence of a table of the inverse function F^{-1}, this will require inverse interpolation in a table of $F(x)$.

Thus, suppose a random observation is to be drawn from the Cauchy distribution with cdf

$$F(x) = \frac{1}{10\pi}\int_{-\infty}^{x} \frac{dt}{1+(t-15)^2/100} = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x-15}{10}\right)+\frac{\pi}{2}\right]$$

Given u, x is determined by the equation $x = 15+10\ \tan\theta$ where $\theta = \pi(u-0.5)$ radians $= (180u-90)$ degrees. If $u = 0.2537$ the corresponding x as obtained from Table 17.7 is given by $15+10\times 0.9770 = 24.77$.

(iv) *Bivariate distribution of the variables X, Y with cdf $F(x, y)$.* Let the cdf of the marginal distribution of X be denoted by $F_1(x)$ and of the conditional distribution of Y given $X = x$ by $F_2(y \mid x)$. A random observation of X, Y is given by x, y where x and y are independent observations from $F_1(x)$ and $F_2(y \mid x)$ respectively chosen in the manner described in (i) to (iii) above.

Thus, suppose a random observation (x, y) is to be drawn from the bivariate normal distribution with the specifications: mean $X = 50$, mean $Y = 75$, variance $X = 100 = (10)^2$, variance $Y = 225 = (15)^2$ and correlation coefficient $= 0.6$. Note that marginally X is normal with mean 50 and variance 100 and conditionally given $X = x$, Y is normal with

mean : $$75+\frac{0.6\times 15}{10}(x-50) = 30+0.9x$$

and variance : $$225[1-(0.6)^2] = 144 = (12)^2.$$

The problem reduces to that of drawing an observation x fron N (50, 10) and then an observation y from N $(30+0.9x, 12)$ which can be done by the procedure explained in (iii) above. To get x, take an observation u from $R(0, 1)$ as explained in (i). If $u = 0.3135$, the corresponding standard normal deviate obtained from Table 3.1 by inverse interpolation, is -0.4860. Hence

$$\frac{x-50}{10} = -0.4860 \quad \text{or} \quad x = 45.140$$

Similarly if $v = 0.5912$ is an independent observation from $R(0, 1)$ with the corresponding standard normal deviate 0.2306, then

$$\frac{y-30-0.9x}{12} = \frac{y-30-40.626}{12} = 0.2306 \quad \text{or} \quad y = 73.393$$

The procedure can be extended to the multivariate normal case with dispersion matrix Σ and mean vector μ.

An alternative and simpler procedure in the special case of the multivariate normal distribution is as follows. First find a matrix A such that $\Sigma = AA'$. If $y' = (y_1, y_2, \ldots, y_p)$ are p independent observations drawn from $N(0, 1)$ as illustrated in (iii) then the observations for the specified multivariate distribution is

$$x = Ay + \mu$$

e. To obtain a random permutation of n digits (elements)

(i) *For $n \leqslant 10$ by using the random permutations given in Table* 19.1

Example: To obtain a random permutation of numbers 1–8 or equivalently of eight letters (symbols) $a, b, c, \ldots, h$.

Choose a serial number at random from 1 to 500 (rows) and select from Table 19.1 the permutation corresponding to the selected row number. Thus if the serial number chosen at random is 232, the permutation to be selected is 5071389264. From this we obtain the permutation of any subset of numbers by omitting the others. In the present problem deleting 0 and 9 we obtain the permutation 57138264 of numbers 1–8.

(ii) *For $n > 10$ using random permutations of Table* 19.1

Example 1 : *To permute numbers* 0–12 *at random.* A random permutation of 0–9 is selected as in (i) above. The positions of numbers 0, 1, ..., 9 are determined by such a selection. We then determine the positions of 10, 11, 12 successively choosing one number at a time. For 10, there are 11 possible positions. It could occur either at the extremities of the selected permutation or in any one of the 9 gaps in between two smaller numbers. The eleven positions could be serially numbered 1–11 and the position of number 10 decided by selecting a number at random from 1–11. Number 11 could then be fitted in an exactly similar manner in one of the 12 possible positions and so on.

Example 2 : *To permute numbers* 0–17 *at random.* One possibility is to repeat the process explained in Example 1, several times, and adding the numbers 10, 11, ... 17 in any succession. A variation of this method is suggested below. The eighteen numbers are divided at random into two sets of nearly equal numbers. This can be easily done by matching the given numbers with the digits in any column of the random number table and taking all the numbers matched with even digits as belonging to the left set and the rest to the right set. If the number in any set exceeds ten, this may be further divided into two sets, the left and right subsets being determined as above. We thus have a number of sets which are already randomly ordered and each of which contains less than 10 numbers. The relative positions of the numbers within each set are determined by permuting these numbers, using the methods in (i) above, independently for each set.

The division of the given numbers into sets may not be a simple operation. We suggest the following general method which uses the random digits of Table 19.1 but not the random permutation of numbers listed in the table.

(ii) *For* $n > 10$ *using a table of random numbers*

Example: *To permute numbers* 1–84 *at random.* One method is to consider two columns of random numbers and note the numbers in the order in which they occur omitting repetitions and the numbers exceeding 84.

A variation of this method due to Rao, which does not omit any number read from the random number table is as follows.

Locate a starting point consisting of a row and two columns of random digits of Table 19.1 for reading two digited numbers. Each number defines a cell in a 10×10 two way table, corresponding to the values of the first and second digits. We put 1 in the cell corresponding to the first number, 2 in the cell of the second number and so on upto 84, as we read the two digited random numbers in the sequence as they occur. The numbers in the cells read out in the order, from left to right in each row and then in the next row and so on, provide a random permutation. If in any particular cell there is more than one number these could be randomly permuted within the cell. The first five numbers corresponding to the random numbers 31, 17, 81, 45, 31... are entered in the chart below to illustrate the method.

second digit

first digit	0	1	2	3	4	5	6	7	8	9
0										
1								2		
2										
3		(1, 5)								
4						4				
5										
6										
7										
8		3								
9										

As it stands we obtain a permutation of numbers 1–5.

$$2\ (1,\ 5),\ 4,\ 3$$

where (1, 5) has to be replaced by a random permutation of the two numbers which can be easily done.

f. Generation of random numbers by coin tossing

This method comes in handy when a random number table is not available. It can also be used to locate a random start in a table of random numbers.

The procedure with an unbiased coin is to toss it a number of times, observe the sequence of heads and tails, and compute a number based on this sequence. A number so obtained is a random number in a certain range. The number of tosses needed to cover a certain range of numbers and the method of conversion of a sequence of heads or tails to a number on a decimal scale are as explained below. Suppose that it is desired, to choose a random number in the range 1–500. First determine the smallest integrer k such that $2^k \geqslant 500$. In this example $k = 9$. Then, toss an unbiased coin k times. Let the observed sequence of heads (1) and tails (0) be

$$0\,0\,1,\ 0\,1\,1,\ 1\,1\,0$$

A random number is then obtained by finding the decimal equivalent of the binary sequence and addiing 1 to it.

The number corresponding to above sequence (or a binary number) is

$$0\times 2^8+0\times 2^7+1\times 2^6+0\times 2^5+1\times 2^4+1\times 2^3+1\times 2^2+1\times 2+0\times 2^0 = 94$$

giving the random mumber $94+1 = 95$.

If the number so obtained is 501 or more, it is rejected and fresh tosses are made. Powers of 2 needed for conversion of sequences to numbers have been given in Table 17.5 (powers of two).

The random number table has 40 columns (on each page) and 500 rows. It is suggested that a random start specified by a row and a column be used in reading the numbers. For this purpose we have to find two numbers one in the range 1–500 representing the rows and another in the range 1–40 representing the columns. The method of generating a random number in the range 1—500 by coin tossing is already explained. To select a random number in the range 1–40, we first choose a number in the range 1–64, which requires 6 tosses, conversion of a six digited binary number and addition of 1 as explained above. If the number chosen is within the range 1–40 it is accepted. If it exceeds 40, it is rejected and the procedure is repeated. This procedure incidentally leads to about 40% rejections. Rejections could be minimised in the following way. If the number obtained exceeds 40 compute its difference (y) from 40. Toss the coin once more and record the result x of toss which is either 0 (tail) or 1 (head). The selected random number is $24x+y$. The number so obtained will always lie in the range 1–48; it is rejected if it exceeds 40, in which case a fresh set of 6 tosses are made and the entire procedure is repeated.

TABLE 19.1. RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
1	7513462980	9787	3792	5241	0556	7070	0786	7431	7157	8539	4118
2	4310765982	4479	1397	8435	3542	8435	6169	7996	3314	1299	1935
3	2731469508	0191	2800	1056	2753	4816	1979	0042	5824	6636	2332
4	6014738925	8710	6903	1347	9332	6962	6786	9875	7565	8683	6490
5	8467523019	4656	5960	0812	5144	5355	3335	4784	7573	3841	4255
6	0689351274	9974	9239	8049	4971	7555	3935	9405	8545	4329	5358
7	0978346251	8493	7128	3654	8976	1901	5496	3453	7539	3255	6742
8	2637098154	6135	6954	3436	3841	9009	3768	9256	3631	9066	7153
9	8761492305	1217	2748	3864	4752	7407	9975	6372	3308	0000	4734
10	6321850749	2623	1282	4389	8889	0764	2328	2140	8843	4986	4413
11	9123075684	1144	5336	4426	9003	6956	9406	8464	8827	3143	4754
12	3047921685	5854	9981	9079	2908	4755	4620	6455	6793	7539	4031
13	9714850236	0615	8188	2812	0270	5733	5339	1175	2919	7343	0477
14	1684205973	3624	0853	3128	7952	2678	3011	7710	9734	6386	8400
15	6132758904	1185	6832	4918	9236	3026	5795	0352	7533	4435	0306
16	0674951832	7391	3210	9540	4085	9324	4892	3962	3883	4538	8286
17	1694508372	7195	1986	6146	0946	5421	8430	2128	7602	5609	7064
18	3168752094	6137	7286	5283	0609	0941	4935	2521	7937	2153	2629
19	4750823961	7401	8099	7482	2210	3662	8253	7507	7809	0094	4401
20	2604381975	0192	9452	7189	9552	7498	0105	8295	9762	7434	3518
21	9708245361	3621	3037	2274	3803	0946	9874	4911	6797	1227	8494
22	3859761402	2661	0047	6628	6199	2526	5631	8334	7668	3994	7439
23	8245139076	8072	5085	3576	4939	0352	7386	7690	7108	6668	8246
24	2409873165	0839	5224	9768	3839	8495	1668	6957	7031	2032	1468
25	2864935170	2354	9266	8034	3813	3648	7825	6156	3605	7796	1645
26	9164078352	9050	6800	0490	3261	7748	3609	1050	0591	3799	2827
27	6053894271	7174	7703	1540	8001	6230	0387	9553	7447	0240	2511
28	2674159083	3465	7017	2278	0357	5800	1048	8382	8800	7608	4325
29	8703615942	8805	1265	5202	6872	3282	5331	5398	1426	2805	2110
30	7410239586	0250	4100	5263	8506	9848	2451	2031	2026	8661	4163
31	8219476035	6088	8366	7751	1577	9534	2458	1886	1522	4161	8726
32	7140532968	8833	3449	3499	4223	2854	6855	4042	1294	1728	5494
33	2709538146	4675	2535	1915	9783	9754	2790	6856	0352	9628	8342
34	5704196823	8999	4993	2922	8842	9904	8442	0105	3308	3320	6361
35	7163482059	1790	8590	5792	0983	3494	0945	4966	2194	9823	2599
36	5017249683	9276	3967	2486	6242	3276	1884	1847	8922	7356	1528
37	3541076298	2965	7991	3777	9303	0536	1517	0570	7212	7593	0566
38	0187326495	6620	4234	8407	6890	6904	8599	5876	2608	7329	6117
39	4791635802	4706	8319	6252	3177	9108	3069	0910	8241	9842	0895
40	6701439825	8395	3882	0259	2092	4885	3434	0879	0000	0790	0735
41	0251784693	3991	3406	0151	2594	9137	9924	2393	7699	6116	0655
42	9452783601	9644	6763	3512	0139	4119	2722	3219	0070	3830	7997
43	8360527149	3658	7813	0207	0357	8225	4497	2435	5121	4776	3611
44	3592076148	5728	1882	9120	7893	3503	8579	9070	1952	8390	5517
45	8456091273	6221	6366	8192	8429	4387	5484	7553	4053	9458	2292
46	7831506249	7635	5248	1750	0868	0173	4989	2300	3916	6732	8284
47	2754319608	4368	3113	5887	8439	0026	1902	4114	3127	5140	6684
48	5207934861	8635	9723	2550	8216	7531	7732	3963	4014	2099	3030
49	6274095183	3304	3254	3936	8361	9771	8255	4592	8808	3803	4010
50	3486157209	9336	5666	1349	1932	7326	2151	1573	3045	8746	8059
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
51	0713629548	8741	2645	3642	5656	7080	7555	2410	6041	7470	0562
52	7543029816	3686	1248	7771	8084	3942	1548	3927	7674	1990	4143
53	3865701942	1670	9305	4099	6266	3502	5194	6177	1622	8869	6783
54	0739415628	2491	1213	1501	1277	3840	1511	7132	2546	2957	8326
55	5894071236	7195	6578	5065	8091	9148	9097	8235	7375	7785	5157
56	1039856472	3260	4487	4306	5689	5971	3809	5141	4733	3276	3419
57	8174205963	4340	5161	2203	1436	7950	9544	7036	4297	1338	1409
58	6358710429	1862	6486	1989	1922	8232	8844	5464	9126	9463	8023
59	8520976314	8561	9331	1291	0346	7278	0365	4273	9956	2792	9918
60	5346107892	1819	6869	6926	6812	7721	3881	6048	8236	0144	7587
61	7541280936	7834	4802	2884	5548	6344	3205	3053	3353	2142	8749
62	1796835024	9336	8624	1342	8268	4123	2800	9291	9617	9601	9472
63	5173426890	9860	4129	5199	2452	7591	9008	2598	1591	3681	9261
64	4360857291	7578	9376	1928	4705	0632	4275	7238	3466	5979	2556
65	1896320754	4345	7810	0617	7257	2879	6235	4505	8289	0552	3300
66	2136498507	5225	0299	7208	5423	7503	7263	0437	0253	8038	5117
67	4685019732	4441	2870	2801	6200	0546	5216	5792	1816	1746	9001
68	7620184953	5276	0963	0021	1488	8719	5283	4018	7415	1731	9880
69	0598267431	2827	0349	3294	6570	4378	5443	3738	8924	2169	4639
70	5718936042	5641	1669	9976	1742	1324	4889	8507	2057	4958	5031
71	5481027369	2942	2259	0736	0154	6117	0113	9627	4820	4419	5788
72	4123096578	9956	9151	0445	1751	9063	8369	9343	8270	5050	0400
73	2567804319	1554	2454	7822	1949	9762	2945	9454	8395	1834	2286
74	1342069875	9645	4489	2281	7385	4674	5252	1454	1582	8154	9824
75	4689357102	6014	2649	5277	5968	3066	5722	3989	3215	1326	6459
76	4580932671	3643	8985	0596	8112	1981	2269	1965	4271	1205	9625
77	5374819026	8821	6444	6587	1157	6305	6856	6878	3239	7638	8178
78	2813450967	1937	9781	6511	3546	9305	0760	6760	2958	9304	3982
79	1730685492	4635	7227	8849	2147	1822	7829	2139	4845	2693	0548
80	2103956478	3250	0102	4089	2463	7465	7254	0119	3201	5409	7813
81	3172046958	1517	5025	6376	6447	2813	2927	4839	1871	8905	7253
82	3851964720	9909	8909	1086	3315	5258	0374	1286	2587	7554	2839
83	0942186357	7856	7048	4321	1759	9625	5353	1993	4504	0291	6843
84	2436785901	2279	8671	8981	1034	1516	7009	5222	8998	9607	8061
85	5063897241	0942	9267	5980	5224	2929	2739	8947	3478	4509	3815
86	5728419630	1497	3941	3923	2222	8174	5799	4694	4466	5799	8261
87	5631824079	9860	8300	9768	6234	2674	4426	6908	0168	3267	3931
88	3879215406	4310	0004	6973	1508	7504	4133	5853	1612	1079	9404
89	1496073582	8670	8162	5768	4631	8650	7852	9727	2108	0341	0145
90	6509742318	1770	1026	6923	4482	0876	7792	7539	7516	4904	0459
91	1863705924	8629	5844	4787	8812	3177	7046	2501	1973	8355	0420
92	0289346715	4303	2321	9487	1278	8670	9876	9746	1016	4839	8978
93	4213805967	8161	6868	2836	4105	8413	3104	2906	1212	6029	4959
94	4927608135	3855	8354	8218	2320	4191	8813	5560	6610	7766	7880
95	0491837625	9849	3393	3314	4959	7959	3144	8546	7348	9163	3794
96	9324157608	7741	7268	4874	8733	7880	1268	7912	8844	7989	1653
97	7438169502	7461	1679	2094	5152	3747	2833	9573	0494	1763	0969
98	4705832691	2773	4477	1491	7922	9026	0902	2755	6727	2186	3504
99	6341590278	0423	5427	7777	0154	0930	5234	7331	5410	4265	1487
100	1274890635	4988	4154	6652	1064	1444	3983	5004	4932	3099	1381
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
101	4918673205	7442	4083	4593	5931	9844	2315	3229	2885	5354	8719
102	6739104582	5968	3047	0472	9141	4220	9038	5181	1297	3073	3897
103	1570428369	9478	9224	8471	0926	0310	9108	1907	0961	5651	8249
104	4538206197	9549	5236	2903	7241	8677	6973	3335	0603	4089	5798
105	2306459718	1763	1972	7035	0591	6761	9105	8156	8160	1915	0154
106	7413096852	8608	3298	1815	7279	0553	0059	6531	4733	3270	3278
107	3740268915	5707	5434	0921	9622	9730	9296	7099	4717	7110	3786
108	6412583790	9012	7487	1766	1747	1157	9468	6225	9161	1447	6750
109	6798421350	2797	0087	0151	4483	3227	0788	5580	8934	6084	0462
110	1530642897	5002	8449	8547	7759	8537	5997	0660	3514	0122	7511
111	3175082694	8053	8671	8718	2844	4898	3540	0545	6249	2134	8217
112	7910485236	8725	9122	8674	5661	1964	7917	5174	8048	1128	5968
113	5670418923	2947	6857	0393	6260	7946	5078	2220	8988	8596	9655
114	5817903642	9545	2086	4017	0290	8043	6378	4422	4235	8116	3074
115	4761892305	3173	3195	4195	4096	8901	0979	6571	3607	0119	2188
116	6351897024	7907	6404	5098	9805	9700	4918	7024	3667	0480	1029
117	5869137240	0660	1290	1481	0170	5976	6817	5761	3709	4728	5168
118	2467509183	8607	5404	9335	2698	9447	4620	1539	0915	6348	2908
119	4321956087	2702	9406	6788	7624	6850	9444	8857	2542	3169	5838
120	9501482736	4092	5306	0210	4018	9752	0865	2948	0117	9410	3168
121	2860345179	1962	3800	0947	1358	1499	2923	1870	4410	1107	6502
122	4605371982	8571	5495	3948	4556	8917	7564	4456	9381	5450	4201
123	5214073689	8583	4396	3916	7627	7870	9243	9996	5474	4545	5176
124	5217690384	6958	6450	6274	4092	7403	4698	6851	7388	6221	7690
125	8026159347	7813	3137	1042	4513	3793	1676	7502	1501	8182	2791
126	3704816295	6202	5081	7437	9792	7482	2452	4566	0063	2908	6206
127	6850249137	3602	3802	9585	0233	1125	7718	1595	1663	9712	1949
128	0321485697	7362	0540	9927	6753	7820	9581	8551	2222	8934	6427
129	0471932865	8594	8665	1428	7110	3650	2606	0638	5831	8923	3804
130	5736482910	8401	8717	6042	3832	0579	6225	3976	9414	6561	1843
131	7450296183	5525	2406	3403	4926	7858	3755	5044	0128	4083	9181
132	3948576102	0435	1737	5504	0171	9765	6487	2778	8138	6776	0208
133	1297035684	7402	4174	6749	1115	1256	4163	1822	5656	9019	0138
134	4817563902	7578	3127	9779	1225	3014	8330	4855	5576	5727	0232
135	5321067849	8080	6396	5680	9512	4867	1879	0353	9354	2673	8744
136	4302596817	2862	2536	6924	2657	4592	0559	8989	0700	8672	9191
137	1243859607	0263	7566	0818	1617	5577	2833	3250	5106	6205	9458
138	4701693258	7025	2207	8947	0037	6266	8351	1668	6054	4681	6656
139	2309415687	4153	6787	3179	5191	8773	6780	8083	0621	1482	1933
140	3497082516	0169	7207	3942	7479	7106	0759	3015	8887	3523	1628
141	3675482091	3477	4956	5529	2512	5832	1931	9402	5236	2853	9537
142	8205936417	7520	9583	1316	0525	4829	1377	3459	1811	9418	1109
143	1534978026	0303	4653	1699	9939	7449	2455	2433	8860	4592	1811
144	1235478096	7735	3302	2025	0631	9888	9188	7751	2595	6431	4409
145	4260537819	8153	2883	5772	4839	9714	1706	9388	9554	1170	3857
146	1938640572	0165	8156	1172	2637	4896	5160	3553	4946	9352	5504
147	7865032491	7849	5240	8311	9400	4421	5264	2153	0571	0809	0978
148	5183726409	6140	2003	7009	2407	8875	4854	8776	2078	7518	3425
149	4908127635	6546	9212	9380	8736	6120	8205	9414	9621	8669	1445
150	6359427108	2471	7972	4706	3986	9176	5669	6598	3196	1572	0455
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
151	6013925478	8094	7747	6006	2536	8856	8171	0291	2603	4675	0779
152	5480271933	3745	6766	7221	9560	7036	4520	4584	5714	8122	5029
153	2108473596	0835	3641	1638	3464	1767	7664	6247	8362	1257	5265
154	1342975608	1601	1143	7272	3988	8356	9477	5870	6425	1725	9792
155	6471893502	7797	4121	9603	5723	4630	5549	0593	5761	7200	7227
156	9253486017	7620	8310	9500	7116	6259	7619	3749	9121	2185	9335
157	8754026139	2096	5270	0793	3950	2722	0925	5792	1040	5806	9636
158	9210846375	6803	7016	1055	6396	7754	3591	2613	5325	7485	2406
159	8409756321	3566	9310	2604	8607	4765	2237	1222	3947	1228	2708
160	7238406591	6428	0086	6245	3247	5707	7847	6217	0857	8229	5609
161	3760451298	8633	2617	9176	9602	4807	7269	6131	8780	3417	7278
162	4502638971	6632	8056	1091	9158	7303	4084	9096	4047	6775	0876
163	4578392610	2612	7936	1453	4812	1742	7128	3636	6561	7522	0359
164	5604298713	9436	1681	0851	3488	8815	5301	5403	5456	0501	4511
165	2756498130	0418	2487	5583	9032	6507	8554	0346	6251	3577	4146
166	8675041392	6853	3757	0171	5943	1145	3434	0188	5665	7779	7179
167	7301642895	8347	7044	4640	6832	2445	4872	7870	2335	2874	9393
168	6482170935	5182	6263	1224	9863	6751	0084	8827	9479	8342	0053
169	4513890726	9215	3992	4874	8082	5959	2861	4574	5813	5903	7161
170	5364219078	5588	3456	9602	5260	6578	8618	0340	3381	7579	6359
171	6291038574	3996	0415	7015	9210	0974	0319	2699	8036	1090	3805
172	6013249578	7346	9400	3292	8165	3206	7035	5227	7340	8515	4225
173	3705129486	8621	4185	6727	2770	1227	3696	6496	4889	2697	3316
174	9840562137	9399	5575	1562	5821	9824	4909	0348	8735	3604	9959
175	4927108356	4334	0347	4893	2025	5590	8126	8571	2532	9355	7563
176	9705642831	8091	0536	6522	5409	1463	0138	0384	6711	2384	0072
177	3015468927	9627	3311	2010	2525	3142	9700	2196	4076	3710	3372
178	5720619483	0086	3501	4916	2511	1274	1775	8324	9646	0611	1048
179	4157826903	3753	0174	7934	3483	9210	9163	4714	7888	3577	6596
180	2068475319	2740	3239	3054	9991	3778	3195	1040	2022	3193	9196
181	1432596087	3919	6871	5685	8147	7310	2080	4196	3375	5700	7967
182	9286347510	4577	7897	2757	5992	7398	7687	8415	1595	9636	4605
183	8714356902	0215	7254	5378	3861	3448	9494	5221	1325	7317	1022
184	8201764593	5807	7948	1774	6836	1786	2392	2820	8533	0629	3771
185	1760524839	1910	9653	1214	3921	5298	8334	2352	7113	2291	9312
186	1652734089	3990	1310	9338	2601	5571	1424	7850	4531	0133	5519
187	1235847609	5967	8941	7987	3335	7579	9735	3042	8409	7053	5364
188	6738210954	5872	1143	9183	6911	2247	1559	4888	7198	9249	1395
189	5428760139	7240	1827	3281	0705	4479	5598	9985	8170	3367	6928
190	1260957438	2268	4227	5844	0700	6907	9668	6670	0097	0686	6311
191	5064871392	8515	1611	1327	6671	2765	0081	0554	3716	9334	3027
192	3819240576	4324	8348	8870	4802	9655	2852	3858	3225	5022	3602
193	1495706328	9053	8503	8222	6850	6100	5973	1522	2690	1396	0632
194	4723098561	5133	7618	3211	0898	5343	9081	8936	0819	9112	2548
195	0324178659	6235	9463	0097	1332	6038	3822	1119	7143	1708	5668
196	7825936014	6048	1376	1589	4274	2920	3521	7661	9435	9257	9276
197	4679013528	6341	0636	3355	7245	4160	1672	2295	4730	0984	6813
198	8537061492	2143	0207	9733	8136	9118	0143	0949	1733	7986	5670
199	5276139408	7336	3277	2135	3300	5287	0134	7104	9359	5069	3893
200	1643875029	2728	6464	4721	8192	5485	7935	4996	3475	9523	5514
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. *(continued).* RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
201	8130726495	6415	5554	3592	8008	9408	2092	9842	3197	1404	1505
202	3012975468	4668	3479	4073	6941	8286	3374	3696	7856	8980	0359
203	9138025467	7592	3903	7895	1113	7646	9201	9081	2630	1617	1188
204	9517246830	2012	1096	2958	4788	4882	1855	8190	9726	6716	1384
205	9061528437	7884	8004	7831	8264	0028	8118	5011	5704	9394	7669
206	7482139650	5510	8160	6173	5655	4415	0147	1091	4426	2843	5578
207	6183572904	4440	0095	4067	9078	6205	7488	1851	3537	7191	0856
208	8370154692	8436	4936	3013	6818	1577	0249	5107	5304	3872	4157
209	5076931248	3740	3172	2775	5781	0318	8932	9220	3784	0501	8375
210	5071428369	1174	3869	9985	4443	1127	7390	1463	8524	2272	4275
211	8361459270	8494	5214	9020	4568	3508	1257	9685	6310	9763	1887
212	9456831072	8792	6689	3521	4407	2017	8527	2230	1851	4023	2258
213	2586413790	0865	4556	4015	0082	1239	7058	1189	3174	0220	1167
214	9758136042	7141	0799	4764	5283	4291	4822	3735	1393	2477	6782
215	5726810943	7185	3986	7047	9210	2791	7610	7264	4771	0548	5172
216	4562183790	3672	8714	8853	9825	5869	6281	2371	1890	9480	2968
217	0592167834	7753	9791	3436	4604	7991	5222	9280	1584	7141	0221
218	5706184329	9332	5082	8900	4209	4117	8644	8712	7337	1689	8793
219	7493825610	0759	2206	4220	2394	4346	8483	6968	2344	1902	0848
220	5801347962	8493	6032	3585	2162	6301	4929	7087	2907	2690	5039
221	5897241630	6776	2659	7323	9619	7727	6460	6745	1051	7662	7513
222	4018976235	4135	7118	4458	1394	0526	5121	2062	0977	7338	5744
223	2613894507	7714	3485	5412	0716	6914	8192	6483	1946	4271	0995
224	9741538026	9777	1915	1183	3177	6568	6698	4649	3899	2691	4413
225	7429108365	7960	4876	8841	3538	4519	0872	5860	8181	5777	0233
226	5024386179	1714	4061	6365	7480	9312	1139	0715	0571	2575	5990
227	9542160738	7460	0288	1075	3483	1041	5427	6457	0985	1657	8742
228	4597312806	0275	8595	0812	9021	4808	8247	0089	7034	8719	5878
229	4209317685	7735	0399	3931	3135	1585	7292	8362	4006	1184	9676
230	2135690874	8661	9964	9969	2444	6095	2003	9320	2837	4397	0297
231	8712509346	1273	7133	4874	1100	7854	4596	6787	8574	6098	5526
232	5071389264	7784	9159	6674	3243	2531	6093	8906	8855	8614	2781
233	0768193425	0707	0067	6433	6058	4381	0146	1186	9913	3668	6347
234	3816295407	9594	8627	5507	2956	6166	7271	9511	5069	1022	9889
235	0549821736	6690	2781	1790	9596	6472	8774	9058	7915	3647	3525
236	3805297164	3476	7990	0690	0043	1357	9568	1541	3726	9223	4385
237	2708491635	9994	1061	7951	3010	6997	4759	0473	2848	7504	6904
238	3582014796	8308	8100	7244	4206	7766	6919	6866	4064	6714	1805
239	2163487590	7260	8057	8779	6368	0601	1872	3160	8731	3646	2789
240	1236509487	4755	3425	1299	7990	8366	1368	3611	8864	1341	9349
241	0528743619	7156	7190	6054	3489	8939	9089	2637	9180	3991	7161
242	3421950687	1469	1763	1918	2547	7708	1900	1665	1860	3078	7851
243	6901875342	1270	4109	9428	0933	1444	7467	1771	3482	1497	6492
244	3986120745	5485	7802	3094	7249	3901	2827	8294	1329	7170	1758
245	9067145238	7123	0850	6297	5479	1416	1837	9305	3749	8541	5161
246	8914302756	2187	4696	2470	7234	4809	5408	3266	6252	5987	5794
247	7489506132	7595	1895	6183	2013	4399	5255	6714	1839	6132	2653
248	0876354192	3021	1523	2005	2009	9631	1274	9902	4203	8312	9572
249	7509348162	3317	8741	2688	9392	0136	9293	7815	1781	1990	4057
250	7439518062	6711	3947	5004	2625	5105	0116	1895	6729	3159	6492
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10–digit permutations	random digits									
251	8793562104	9877	3443	1246	8363	6403	4920	6437	3957	2660	7523
252	5481273690	9899	1462	1924	4346	3669	4836	9199	8824	6269	6068
253	4510726983	4491	5402	1718	6410	4123	6764	5759	4814	2773	7641
254	0351986427	1045	4241	0208	5923	4148	9843	9628	4909	9109	9712
255	3501247968	0558	5018	1539	2251	0689	4033	5222	0394	4654	3795
256	4572368910	1676	8914	6220	0399	5738	3630	1481	6205	2026	6702
257	5236974018	3322	9745	9596	9208	7021	1663	5240	0627	4177	8243
258	0179326854	7913	3397	8773	9562	6671	1993	0239	6832	0975	7985
259	8296540317	8760	4120	5060	9597	4501	8388	6597	6568	4537	3542
260	7541308692	3681	7110	9412	8239	2749	3100	4266	6170	2118	6077
261	5279640381	3261	1462	0579	3234	6068	7770	3082	3200	9298	1427
262	2510874963	8028	5433	8504	2842	5338	3347	2322	5085	8291	9086
263	8354201679	1271	2976	8910	0356	6389	8537	5013	4733	9121	2195
264	5812609743	2957	9594	5194	7035	8345	4088	4932	1624	2997	6593
265	4358107269	0834	0997	5573	8671	7025	2419	9457	9265	0248	8799
266	5026917384	9271	7247	0360	7287	1971	2242	2839	6233	0244	8140
267	1462853097	2564	9682	0609	0294	8783	2764	4985	0427	7480	3724
268	2049831567	1109	2612	3772	4873	2686	7523	2620	5142	2131	0525
269	5317986420	0398	8679	4741	9834	3643	1471	6154	8734	5630	6651
270	9201463578	9615	4280	5324	9330	6797	6282	9107	1458	3012	6128
271	0192564378	1604	0876	5340	9493	6324	2798	3666	8417	3691	1194
272	6582703914	0979	5483	9569	8397	4437	0777	0800	8645	2094	9569
273	6048971532	2177	1316	0091	9792	2661	9132	1132	4763	6277	1510
274	8147395602	3193	8126	9538	3418	4336	9254	8381	9545	4057	6320
275	6539182047	7517	5274	3499	3961	8029	8727	0535	9501	0700	2846
276	1286349057	9712	9515	4770	9913	5808	8769	0877	4004	4712	8363
277	5147683209	9109	0961	8022	6694	2960	9755	5054	3854	6245	9032
278	8352469071	5630	6764	9685	2941	8903	4099	0980	9857	7134	6406
279	4785239160	6319	6648	4706	4820	1422	5725	5686	6028	2061	2470
280	0214638579	8989	2630	1052	6555	5278	1774	5635	4559	4206	7153
281	4827395016	2394	0140	1210	8008	6250	4190	7221	3080	6689	8212
282	8369215074	5100	4052	7384	4677	3943	1907	3168	1277	7266	4585
283	5198642307	1389	6494	7415	2106	5428	4678	0556	4776	6499	3480
284	2078369451	3826	3510	2476	7985	0711	1038	9273	7722	5286	0842
285	2438791605	1238	9343	1109	9487	2400	6970	0625	3044	2437	5701
286	4769218503	1080	4414	8662	8020	2884	7692	8325	9513	4957	6704
287	1498652307	7137	0030	3270	4046	4393	0971	9503	1827	2830	3629
288	9572486031	0840	7292	2063	4966	0385	3357	8639	4840	8063	8433
289	7835910642	3978	2534	4276	9021	6937	8509	2543	0835	9268	9159
290	2680534719	6697	8084	8397	4451	0046	3073	8916	7666	5480	7462
291	1247580693	2861	1319	6439	3530	1042	8177	6019	0439	9395	7133
292	5867390412	5813	8664	2493	0798	9076	8985	4252	7852	2174	4663
293	9815246037	4251	0630	5910	9077	7568	8906	9793	8706	2865	1230
294	8045619327	9041	7830	0919	4651	4484	3545	6344	1245	1477	1337
295	2056978143	2893	1125	2539	2960	4321	3832	3424	1018	3347	9543
296	5304629718	4527	7618	8173	0665	6557	9782	5576	2138	5580	0332
297	5768423901	2755	6055	5247	9461	8787	4995	4248	2740	0081	3388
298	8953204761	0593	1552	0641	3472	4616	8097	1890	1516	4423	7168
299	4829063715	8776	1623	7215	6291	0520	7414	4329	7957	6609	1446
300	6450239871	0659	6683	8255	3657	1215	3650	3049	7781	9327	3137
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
301	9257830146	3436	6833	5809	9169	5081	5655	6567	8793	6830	1332
302	6473180592	6133	4454	2675	3558	7624	5736	2184	4557	0496	8547
303	0295431786	9853	3890	5535	3045	9830	5455	8218	9090	7266	4784
304	0564329187	5807	5692	6971	6162	6751	5001	5533	2386	0004	2855
305	8976321045	6291	0924	1298	7386	5856	2167	8299	9314	0333	8803
306	6245908713	4725	9516	8555	0379	7746	9647	2010	0979	7115	6653
307	2956403187	7697	6486	3720	6191	3552	1081	6141	7613	5455	3731
308	8275036419	3497	2271	9641	0304	4425	6776	1205	2953	5669	1056
309	7934508612	8940	4765	1641	0606	4970	7582	7991	6480	2946	5190
310	1290578364	1122	6364	5264	1267	4027	4749	0338	8406	1213	5355
311	4328065971	4333	0625	3947	1373	6372	9036	7046	4325	3491	8989
312	9537082164	7685	1550	0853	4276	1572	9348	6893	2113	8285	9195
313	4369507182	0592	8341	4430	0496	9613	2643	6442	0870	5449	8560
314	7139824560	3506	0774	0447	7461	4459	0866	1698	0184	4975	5447
315	1947658320	8368	2507	3565	4243	6667	8324	3063	8809	4248	1190
316	4265801793	2630	1112	6680	4863	6813	4149	8325	2271	1963	9569
317	6159078324	3883	3897	1848	8150	8184	1133	6088	3641	6785	0658
318	0347192568	1123	3943	5248	0635	9265	4052	1509	1280	0953	9107
319	6072148593	1167	9827	4101	4496	1254	6814	2479	5924	5071	1244
320	1769802354	7831	0877	3806	9734	3801	1651	7169	3974	1725	9709
321	3465701289	2487	9756	9886	6776	9426	0820	3741	5427	5293	3223
322	9140852736	1245	3875	9816	8400	2938	2530	0158	5267	4639	5428
323	0267394581	5309	4806	3176	8397	5758	2503	1567	5740	2577	8899
324	1768942035	7109	0702	4179	0438	5234	9480	9777	2858	4391	0979
325	9325401867	8716	7177	3386	7643	6555	8665	0768	4409	3647	9286
326	6514803927	9499	5280	5150	2724	6482	6362	1566	2469	9704	8165
327	0769524183	3125	4552	6044	0222	7520	1521	8205	0599	5167	1654
328	4018637529	3788	6257	0632	0693	2263	5290	0511	0229	5951	6808
329	1864793052	2242	2143	8724	1212	9485	3985	7280	0130	7791	6272
330	5139064782	0900	4364	6429	8573	9904	2269	6405	9459	3088	6903
331	2145798063	7909	4528	8772	1876	2113	4781	8678	4873	2061	1835
332	1738294506	0379	2073	2680	8258	6275	7149	6858	4578	5932	9582
333	7095432681	0780	6661	0277	0998	0432	8941	8946	9784	6693	2491
334	8312670594	8478	8093	6990	2417	0290	5771	1304	3306	8825	5937
335	8763104295	2519	7869	9035	4282	0307	7516	2340	1190	8440	6551
336	3570694281	2472	0823	6188	3303	0490	9486	2896	0821	5999	3697
337	5062471893	8418	5411	9245	0857	3059	6689	6523	8386	6674	7081
338	5842173960	8293	5709	4120	5530	8864	0511	5593	1633	4788	1001
339	6524319807	9260	1416	2171	0525	6016	9430	2828	6877	2570	4049
340	7418620359	6568	1568	4160	0429	3488	3741	3311	3733	7882	6985
341	4538927160	6694	5994	7517	1339	6812	4139	6938	8098	6140	2013
342	0426371958	2273	6882	2673	6903	4044	3064	6738	7554	7734	7899
343	3142598607	6364	5762	0322	2592	3452	9002	0264	6009	1311	5873
344	5297804631	6696	1759	0563	8104	5055	4078	2516	1631	5859	1331
345	4926530817	3431	2522	2206	3938	7860	1886	1229	7734	3283	8487
346	3701645982	4842	3765	3484	2337	0587	9885	8568	3162	3028	7091
347	7402913658	8295	9315	5892	6981	4141	1606	1411	3196	9428	3300
348	6071843925	4925	4677	8547	5258	7274	2471	4559	6581	8232	7405
349	8936710452	5439	0994	3794	8444	1043	4629	5975	3340	3793	6060
350	0694137852	2031	0283	3320	1595	7953	2695	0399	9793	6114	2091
Column no :		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
351	6743059218	0883	2339	1363	4219	0189	4453	0806	1970	4130	7998
352	9614580723	4634	6385	8760	3555	0567	8815	4700	5092	0231	5757
353	8527140693	5432	9770	2781	6469	7152	0256	6137	0458	0968	9610
354	2536719804	2317	5966	3861	0210	8610	5155	9252	4425	7449	0449
355	3576492108	6836	2472	0385	4924	0569	6486	0819	9121	8586	9478
356	3601874592	9358	5197	4910	0263	2372	6446	0252	0383	6518	0707
357	7839402615	5936	9276	7805	3690	7473	5954	3164	3482	1845	7686
358	5780436192	4306	9165	6438	6777	4671	2360	3382	2686	8767	6827
359	9502813764	5951	7275	3713	5951	1452	1986	5034	0518	9314	7164
360	1754268039	2108	6157	6254	7483	2407	8609	2114	4095	2456	8169
361	5873062914	9566	6198	4546	8964	4473	5657	9152	3956	6235	9991
362	2087431596	3981	3873	6448	0871	2825	7693	9304	9016	5871	9251
363	5679123084	8696	2811	5419	9481	4498	1718	7871	1245	7915	2534
364	3046957821	1433	1167	7332	0970	0159	1218	4679	9568	5533	8206
365	9632851740	2141	6763	3519	7475	5991	8210	6588	5652	2636	7328
366	1237960854	5445	6443	2930	1322	7296	4063	9397	4389	1295	3782
367	0921354876	1339	4168	2508	0980	4184	7238	1406	9956	8366	9846
368	4198705362	0948	6094	9141	8128	5545	9938	2129	7718	3561	2918
369	5238674910	4252	3165	2934	4966	8313	0339	3724	9779	3113	9747
370	0426531798	1898	4922	5411	9237	4511	6360	1905	9126	8473	8258
371	8037695214	4014	3915	9924	2185	0045	5419	3618	0388	8833	7820
372	2156893407	2177	3510	0681	6548	5318	7449	5776	5519	2420	5532
373	8621453970	6625	0747	4812	5649	1408	3724	3681	1637	8352	4305
374	6748152039	8271	1876	2939	1452	3071	0649	4840	9228	5237	5551
375	0712895436	5745	1306	9341	2202	9409	3255	7968	6629	6267	4004
376	6931725084	6164	6330	1234	4065	0816	7058	6369	1947	7346	4723
377	0814976352	9956	5248	7969	9843	3265	5024	0971	4740	3295	2557
378	5712069384	9811	9364	8786	4365	7833	0898	5798	9136	3829	5329
379	5204968731	7346	9293	7714	6558	1103	9861	4270	3645	0912	3498
380	2509681734	8061	5526	9875	6795	9549	2156	0845	0166	5267	1713
381	0768253419	8425	0589	3180	4949	9893	8201	4108	6655	5819	1862
382	6397021458	6464	9513	4697	4312	8602	7950	6790	1419	0407	6701
383	5284613709	5382	7915	3116	5410	2990	9157	6348	3856	6925	0790
384	3471856209	1933	3542	9212	3714	7075	1858	9857	1252	0681	5627
385	3765091824	6426	5146	8050	5391	0055	6736	6866	0829	7983	3239
386	2561397840	6984	3252	3254	1512	5402	0137	3837	1293	9329	1218
387	1054823796	9080	7780	2689	8744	2374	6620	2019	2652	1163	7777
388	4957182603	5583	3674	4040	8915	2860	9783	2497	6507	5084	8877
389	9146237508	8578	8170	3723	8433	3395	2329	7783	7511	7075	1126
390	3579641802	3899	0413	0663	3896	2100	3516	7169	0934	8257	9755
391	5106497283	9372	7493	9462	3932	7468	3383	4358	7937	2542	5480
392	5312968740	4747	1794	4498	1693	0955	5373	5400	5226	4811	0379
393	9876251403	3545	6861	4232	3952	9316	1867	0537	2144	1034	9889
394	0526849137	0836	9910	8303	7618	9262	7540	1802	7089	7172	0442
395	4579286031	9742	4735	1085	9715	2103	5485	3740	4117	2786	5815
396	6831592704	9890	5980	2778	5956	6128	2384	8501	3302	7232	6363
397	0652817439	5960	4185	7079	8917	2378	6868	6472	9093	8609	4008
398	7395084126	9017	3136	4463	4174	8453	5045	4925	7889	7188	6990
399	5621834097	8520	7719	6078	0293	0525	7426	8334	2367	5490	4960
400	9372586410	1436	3124	0072	5146	8555	7584	8382	1378	3848	7323
Column no :		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1. (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10–digit permutations	random digits									
401	0615289347	5697	7118	6204	9111	6389	4456	9293	9662	3299	2935
402	7056289413	7108	5084	6610	1034	9230	8928	3074	2424	5437	5243
403	7509612483	1624	2174	9153	1805	5961	7497	3182	7768	9345	4093
404	4273198560	4342	5983	2381	8327	6084	8620	4531	1922	2839	1920
405	9716038425	0764	8315	5133	3907	1034	1176	9280	3858	6379	0076
406	7394256801	8134	2608	5206	0297	0229	2752	8346	7236	2162	7056
407	1265834970	0446	9907	3887	8015	3138	8184	1222	1401	4968	9433
408	9625780314	0168	0763	4485	0308	6621	7216	8142	9086	6067	3473
409	6809374251	8910	0950	4720	8350	9523	9455	4871	5453	6876	8304
410	2064398517	6313	2963	7027	1611	2298	0888	8981	4069	2411	3119
411	2053147968	1868	1611	5833	4766	7364	8600	9629	6325	1391	0901
412	9231647508	1908	2354	8598	7534	8173	3789	2529	4937	9692	8363
413	7039825614	5757	4234	1566	2521	0011	3478	7744	5426	9996	7460
414	7430658129	4894	8977	4166	5460	6695	4673	7659	2005	6656	2091
415	6425931807	9972	7151	7092	5335	8480	8794	6615	9080	6724	3734
416	8104673952	0397	1612	5516	8463	3357	1826	2352	3770	5699	1631
417	7892364051	6874	2700	2916	1135	3831	6614	6820	6405	0768	2614
418	4318095627	1790	4160	9134	8509	8890	6120	0731	6922	8288	1982
419	4970618523	5409	9981	9730	2675	7209	1940	6072	3082	1266	3850
420	0786194325	1386	9019	0220	1364	5470	4172	1296	6836	9179	2149
421	7490862153	9062	3253	1590	7867	7538	6262	2408	3808	7447	0049
422	9780652413	4926	5410	2930	7402	9141	9168	8655	0806	7715	1242
423	8921643705	5526	3988	7609	8228	8349	3680	0758	1432	9650	5813
424	1435862709	7703	8807	5387	1303	6734	6009	2442	0457	2930	5691
425	9417608352	4837	2243	4989	0616	6385	0136	3689	4829	0446	0570
426	4627905813	6024	8888	2384	8344	9908	5510	9386	3507	9794	9938
427	9102534867	6815	9711	4002	3802	4827	5707	4947	0252	5829	9415
428	4397061582	0225	9718	8245	5335	1690	2306	5836	3721	2226	1627
429	0874635921	1830	9355	6971	2875	2867	6622	4091	7390	1059	8368
430	1267450389	2932	7067	1308	4371	3010	3692	5038	2395	6062	8973
431	0943251678	6390	0765	0975	4201	5564	5937	6244	5111	1524	2020
432	0147256938	6026	2262	4871	9986	7207	3039	8020	9710	8848	4973
433	1986537024	5202	3537	5017	2359	3402	6282	2138	7115	5463	6118
434	8351624790	3397	7794	8411	4512	9632	1542	6757	6911	2985	4853
435	7194582630	3314	9485	5407	3639	2300	2125	9724	1079	0774	1401
436	0659381742	4167	2485	7145	1215	6515	3804	6166	3957	6560	3638
437	5947362180	4852	4039	9145	9178	9429	1919	5290	5257	1535	2001
438	8327950164	6808	4890	2380	2370	4759	5391	6534	9283	6629	7265
439	2856309471	0176	6242	6360	1762	5903	5237	1680	3564	1463	4713
440	7968053142	5444	8089	0748	1112	8211	5432	5547	9680	9872	4939
441	2865349710	1736	7743	2822	7668	3971	3550	9693	8756	7296	9464
442	0694528173	4476	1717	7629	8040	5665	4396	2086	9231	0693	7469
443	3216047589	8918	8308	9085	8062	7813	9579	6144	3710	3853	8646
444	9641583072	5306	1670	9035	4119	0977	4199	5951	1147	8236	5327
445	0589162734	1820	3765	7173	1487	9696	2143	9768	0264	8344	3024
446	7412803569	2232	2463	1604	7905	5999	7615	1020	3755	0686	2767
447	4213806579	2232	9088	3557	7313	1870	0993	4117	1039	6510	9007
448	3296517840	9374	8584	7212	1940	4743	9530	5993	2885	2761	0779
449	3197026548	4971	2715	8838	3135	7893	5168	3081	9046	9998	1106
450	8764059321	8349	3916	1368	0174	1943	9582	6585	6581	0050	5369
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 19.1, (*continued*). RANDOM DIGITS AND DIGIT PERMUTATIONS

row number	10-digit permutations	random digits									
451	1069872453	0362	4799	7572	1970	2923	5912	2303	9270	9739	9041
452	8714036529	2207	4051	9770	4367	8588	9920	5245	0122	5029	5341
453	0726948531	3541	3275	2045	1534	9632	1028	3461	6191	0036	8804
454	1436258790	6378	8747	5602	3128	6345	3973	7275	3768	1449	7837
455	4130987562	1458	1044	9041	9180	6759	0544	6142	7778	8791	8487
456	9840623751	5085	1982	8691	3020	9502	2141	1459	2843	6297	6396
457	2956813074	4518	6537	3071	5227	6196	8352	6297	3905	8918	0826
458	9768350214	6168	5798	1011	6378	6288	8205	3058	5566	5316	8956
459	4713025869	1670	7007	3793	6476	5471	3584	1395	9388	7834	1015
460	5942718306	8329	3831	0731	1917	7710	6905	1885	9986	9578	0338
461	6219387504	1509	7055	5175	5973	7101	8379	4071	9467	1654	4314
462	2958463710	0238	0034	3684	9499	8442	7914	9244	1841	7884	0810
463	4572836091	6673	5806	9370	4519	5256	5061	4908	5691	1424	9636
464	3791548602	7803	0356	3757	7681	3067	8106	0953	8612	7585	2735
465	5960873241	1939	2795	6221	2694	4655	1459	4597	4338	7159	5030
466	5390648721	8765	6905	8958	6987	6878	2380	9707	4807	5051	7022
467	7168502943	6096	7678	5107	8749	3109	2760	4298	8961	3707	1076
468	1943056872	9608	6691	2921	0658	8838	5317	8984	5621	8445	2404
469	6750123849	3725	9751	3433	4341	6965	6050	4132	4739	8388	4777
470	9182754603	6509	0092	3703	0920	0783	0235	3804	2352	2730	2590
471	4259761308	4746	3350	0860	4264	6950	5255	1742	0372	9864	0442
472	6273140598	7259	3378	0985	6983	4750	4446	3526	7085	9876	8324
473	8034129657	0087	4614	9579	1152	5817	3089	9856	3208	9753	8233
474	4073918562	3921	2227	0975	5869	6486	6217	3178	9780	1432	0450
475	2814760593	0430	1184	7306	5822	2892	1993	9895	2603	2430	6093
476	2105934786	5712	4565	4363	2117	0196	2209	4340	2617	5291	8696
477	7641095382	7644	3565	1413	6722	9198	4226	9249	1065	1781	0353
478	7506491283	1915	6992	1157	6470	0165	0341	5839	7973	6543	8881
479	3804196725	6200	7683	0763	1671	0999	2475	7619	0871	1160	9157
480	4608721953	5169	9227	9357	5554	8989	2002	9518	2695	7331	0751
481	8265937410	5204	2143	3487	6244	8168	9846	4364	8984	6648	3560
482	4387196025	3694	6061	1818	2835	6261	4441	6424	7983	9536	0973
483	3584960172	2567	1562	2597	1894	6180	2082	0067	1954	3377	0155
484	1840537629	4335	6678	9377	1391	7460	5914	5452	6939	1890	4383
485	3926840517	3995	1086	5203	2220	5949	6201	5737	3540	3843	7760
486	1387095462	6754	8246	0606	8769	3753	5594	1562	4954	2214	2168
487	2965083174	2008	9669	2075	5664	9584	0312	4676	4402	0603	5566
488	0263594781	3761	2928	6770	9818	2112	8949	4369	8235	3350	5331
489	3184726509	6516	2412	4496	8543	1664	0467	1346	2442	7237	9439
490	9630785241	0644	1902	0678	6897	1244	0429	7083	0771	3267	0563
491	6520479138	9183	7513	8028	0193	9555	8084	2841	8311	4207	7812
492	1034296587	8858	9502	7453	0244	8500	0830	1759	7854	8488	7616
493	4163827950	0083	1818	7118	2553	0655	9201	9554	2362	1364	6135
494	3815496720	4506	1474	1880	2607	5438	4356	7262	4599	3866	2355
495	4891056237	3138	9559	3138	3560	7246	6791	0573	8441	7713	0626
496	0876459312	8940	3713	7631	3874	1347	0713	3889	5376	1262	2340
497	7980346215	5717	5274	0511	2848	8412	3412	4698	3715	4137	1942
498	2514987603	6855	4133	3523	9627	8923	0801	3041	8030	7719	6272
499	9024851763	4094	6732	0816	5004	6964	1310	6449	4474	3448	8346
500	4562198703	1615	5138	9856	2044	0410	5524	0736	9067	4551	4130
Column no.:		1–4	5–8	9–12	13–16	17–20	21–24	25–28	29–32	33–36	37–40

TABLE 20.1. MATHEMATICAL, PHYSICAL AND OTHER CONSTANTS

Mathematical Constants

$\pi = 3.14159\ 26535\ 89793$	$\sqrt{\pi} = 1.77245\ 38509\ 05516$	$e = 2.71828\ 18284\ 59045$
$\pi^2 = 9.86960\ 44010\ 89359$	$\sqrt{2\pi} = 2.50662\ 82746\ 31001$	$\frac{1}{e} = 0.36787\ 94411\ 71442$
$\frac{1}{\pi} = 0.31830\ 98861\ 83791$	$\frac{1}{\sqrt{\pi}} = 0.56418\ 95835\ 47756$	$\log_e 10 = 2.30258\ 50929\ 94046$
$\frac{1}{\pi^2} = 0.10132\ 11836\ 42338$	$\frac{1}{\sqrt{2\pi}} = 0.39894\ 22804\ 01433$	$\log_{10} e = 0.43429\ 44819\ 03252$
$\log_{10}\pi = 0.49714\ 98726\ 94134$	$\log_e \pi = 1.14472\ 98858\ 49400$	$\gamma = 0.57721\ 56649\ 01533$ (Euler's constant)
$\sqrt{2} = 1.41421\ 35623\ 73095$	$\sqrt{3} = 1.73205\ 08075\ 68877$	$\sqrt{10} = 3.16227\ 76601\ 68379$

1 radian = 57.29577 95130 82321 degrees   1 degree = 0.01745 32925 19943 radians.

Numeration

Indian			UK	USA
Sata = 10^2	Koti = 10^7	Mahapadma = 10^{12}	Hundred = 10^2	Hundred = 10^2
Sahasra = 10^3	Arbuda = 10^8	Sanku = 10^{13}	Thousand = 10^3	Thousand = 10^3
Ayuta = 10^4	Abja = 10^9	Jaladhi = 10^{14}	Million = 10^6	Million = 10^6
Laksha = 10^5	Kharva = 10^{10}	Antya = 10^{15}	Billion = 10^{12}	Billion = 10^9
Niyuta = 10^6	Nikharva = 10^{11}	Madhya = 10^{16}	Trillion = 10^{18}	Trillion = 10^{12}
		Parardha = 10^{17}		

Prefixes

Prefix	Value	Prefix	Value	Prefix	Value
Micromicro or Pico	10^{-12}	Centi	10^{-2}	Kilo	10^3
Millimicro or Nano	10^{-9}	Deci	10^{-1}	Mega	10^6
Micro	10^{-6}	Deka	10	Kilomega or Giga	10^9
Milli	10^{-3}	Hecto	10^2	Megamega or Tera	10^{12}

Basic Units of Measurements

(Exact conversion factors are indicated in bold face)

Length

(Symbol : m = metre, dm = decimetre, dkm = dekametre, hm = hectametre etc.)

British Units	Metric Units	Conversion Factors
	10^7 Å = 1 mm*	
12 inches = 1 foot	10 mm = 1 cm	1 inch = **0.0254** m
3 feet = 1 yard	10 cm = 1 dm	1 foot = **0.3048** m
$5\frac{1}{2}$ yards = 1 rod, pole or perch	10 dm = 1 m	1 yard = **0.9144** m
4 poles = 1 chain	10 m = 1 dkm	1 mile = **1.609344** km
10 chains = 1 furlong	10 dkm = 1 hm	1 nautical mile† = **1.853184** km
8 furlongs = 1 mile	10 hm = 1 km	1 metre = 39.370079 in; 3.28084 ft; 1.093613 yd
6 feet = 1 fathom		1 km = 0.6213712 miles
120 fathoms = 1 cable length	(1 knot = 1 nautical mile per hour)	
6080 feet = 1 nautical mile		

1 metre is (very nearly) 10^{-7} of the distance from the pole to the equator.

*Angstrom unit (Å) is used to measure the wave length of light and is equal to 10^{-10} m

† International nautical mile = 1.852 km.

TABLE 20.1. (*continued*). MATHEMATICAL, PHYSICAL AND OTHER CONSTANTS

Area

British Units	**Metric Units**	**Conversion Factors**
144 sq inches = 1 sq foot	100 sq mm = 1 cm^2	1 sq yd = 0.836127 m^2
9 sq feet = 1 sq yard	100 sq cm = 1 dm^2	1 sq ft = 0.092903 m^2
$30\frac{1}{4}$ sq yards = 1 sq rod, pole or perch	100 sq dm = 1 m^2	1 sq in = 645.16 mm^2
40 sq rods = 1 rood	100 sq m = 1 are	1 sq m = 1.19599 sq yds
4 roods = 1 acre	100 ares = 1 hectares	= 10.76391 sq ft
640 acres = 1 sq mile	100 hectares = 1 km^2	1 sq cm = 0.1550003 sq in
		1 sq mile = 2.589988 km^2
(1 hectare = 2.471054 acres)		1 sq km = 0.386102 sq miles

Volume

British Units	**Metric Units**	**Conversion Factors**
1728 cu inches = 1 cu foot	1000 cu mm = 1 cu cm	1 cu ft = .0283168 m^3
27 cu feet = 1 cu yard	1000 cu cm = 1 cu dm	1 cu in = $1.63871 \times 10^{-5} m^3$
	1000 cu dm = 1 cu m	1 cu dm = 0.0353147 cu ft
		1 cu cm = 0.0610237 cu in

Capacity

(Abbreviations : l = litre,** dl = decilitre, dkl = dekalitre etc.)

British Units	**USA**	**Conversion Factors**
(Liquid)	(Liquid)	(Liquid)
60 minims = 1 drachm	60 minims = 1 dram	1 pint (Br.) = 0.568261 dm^3
8 drachms = 1 ounce	8 drams = 1 ounce	1 pint (USA) = 0.473176 dm^3
5 ounces = 1 gill	4 ounces = 1 gill	1 gallon (Br.) = 4.54609 dm^3
4 gills = 1 pint	4 gills = 1 pint	1 gallon (USA) = 3.78541 dm^3
2 pints = 1 quart	2 pints = 1 quart	1 gallon (Br.) = 1.20095 gallons (USA)*
4 quarts = 1 gallon	4 quarts = 1 gallon	1 gallon (USA) = 0.832674 gallons (Br.)*
(Dry)	(Dry)	1 ounce (Br.) = 0.960760 ounces (USA)†
2 gallons = 1 peck	2 pints = 1 quart	1 ounce (USA) = 1.04084 ounces (Br.)†
4 pecks = 1 bushel	8 quarts = 1 peck	1 cu dm = 1.75975 pints (Br.); 2.11338 pints (USA); 0.219969 gallons (Br.); 0.264172 gallons (USA)
8 bushels = 1 quarter	4 pecks = 1 bushel	
Metric Units		(Dry)
10 ml = 1 cl		1 bushel (Br.) = 36.3687 dm^3
10 cl = 1 dl		1 bushel (USA) = 35.2391 dm^3
10 dl = 1 l		1 bushel (Br.) = 1.03206 bushels (USA)
10 l = 1 dkl		1 bushel (USA) = 0.968939 bushels (Br.)
10 dkl = 1 hl		1 cu dm = 0.0274962 bushels (Br.)
10 hl = 1 kl		= 0.0283776 bushels (USA)

* Also true for quarts, pints and gills

† Also true for drachms (drams) and minims

**At the 12th General Conference on Weights and Measures held in 1964, the earlier definition of litre (which was equal to 1000.028 cu cm) was annulled and it was declared that the word litre may be used as a special name given to cubic decimetre.

TABLE 20.1. (*continued*). MATHEMATICAL, PHYSICAL AND OTHER CONSTANTS

Weights

(Abbreviations :
kg = kilogram, cg = centigram, dg = decigram, dkg = dekagram, hg = hectagram, cwt = hundred weight)

British Units	Metric Units	Conversion Factors
Avoirdupois (av), General System		
16 drams = 1 ounce	10 mg = 1 cg	1 grain = **0.06479891** g
16 ounces = 1 pound	10 cg = 1 dg	1 ounce (ap. or t.) = 31.10348 g
28 pounds = 1 quarter	10 dg = 1 g	1 ounce (av.) = 28.349523 g
4 quarters = 1 cwt	10 g = 1 dkg	1 gram = 15.43236 grains = 0.03215075 oz (ap/t)
20 cwt = 1 ton*	10 dkg = 1 hg	= 0.03527396 oz (av.)
14 pounds = 1 stone	10 hg = 1 kg	1 pound (ap. or t.) = 0.3732417 kg
Apothecary Units (ap), Drugs	100 kg = 1 quintal	1 pound (av.) = **0.45359237** kg
20 grains or minims = 1 scruple	1000 kg = 1 tonne (metric)	1 kg = 2.679229 lb (ap./t.) = 2.2046226 lb (av.)
3 scruples = 1 drachm	200 mg = 1 carat	1 cwt = 50.80235 kg
8 drachms = 1 ounce	**USA**	1 quintal = 1.968413 cwt
12 ounces = 1 pound	1 short ton = 2000 pounds (av.) s. ton	1 ton = 1.0160469 m. tonne
= 5760 grains		1 slug = 14.5939 kg
Troy Units (t) Precious metals	1 long ton = 2240 pounds (av.)	1 ton (short) = 0.90718 m. tonne
480 grains† = 1 ounce	1 kip = 1000 pounds (av.)	1 m. tonne = 0.9842065 ton = 1.1023113 s. ton
12 ounces = 1 pound		

* 1 short ton (USA) = 2000 pounds (av.) = 2 kips

† The grain is the same whether it is avoirdupois, troy or apothecaries' weight.

Physical Constants

1 knot (international) = 101.269 ft/min. = 1.68781 ft/sec. = 1.15078 miles/hr.
1 micron = 10^{-3} mm.
Ionic (electronic) charge (e) = 4.80×10^{-10} E.S.U. Mass of electron (m_0) = 9.1085×10^{-28} g.

Mass of hydrogen atom = 1.673×10^{-24} g.
Gas constant (R) = 8.3170×10^{7} erg/degree/gram mole (physical scale) = 8.315×10^{7} (chemical scale).
Avogadro's number = 6.02486×10^{23} per gram mole (physical scale) = 6.02332×10^{23} (chemical scale).
Planck's constant (h) = 6.62517×10^{-27} erg-sec. Boltzmann constant (k) = 1.38044×10^{-16} erg/degree.

Density of Mercury at 0°C = 13.5955 g/cu cm. Density of water, maximum at 3.98°C = 0.999973 g/cu cm.
Density of air, 0°C and 760 mm pressure = 1.2929 g/l.

Velocity of sound in dry air, 0°C = 331.36 m/sec. = 1087.1 ft/sec.
Velocity of light in vacuum = 2.997929×10^{10} cm/sec.

Heat of fusion of water at 0°C = 79.71 cal./g. Heat of vapourisation of water at 100°C = 539.55 cal./g.
Electrochemical equivalent of silver = 0.001118 g/sec. international ampere.
Absolute wave length of red cadmium light in air, 15°C, 760 mm pressure, = 6438.4696 angstrom units.
Wave length of orange-red line of krypton, 86 = 6057.802 Å.

Decibel is a measure of sound intensity in logarithmic scale. Zero decibel loudness level corresponds to an intensity (J_0) of 10^{-16} Watt/sq cm or 10^{-9} erg/cm^2/sec. An intensity J expressed in decibel units is 10 $\log_{10}$ (J/J_0).

The Earth

Polar radius = 6357 km = 3951 miles, Equartorial radius = 6378 km = 3964 miles

Mean radius = 6371 km = 3960 miles

Flattening = 0.003367

Circumference = 24,920 miles

1° of latitude at equator = 110.5 km = 68.70 miles

1° of latitude at poles = 111.7 km = 69.41 miles

1° of longitude at equator = 111.3 km = 69.17 miles

Inclination of equator to ecliptic = 23°27′

Surface area = 5.101×10^8 km^2, Volume = 1.083×10^{12} km^3

Mass = 5.980×10^{27} g = 6.586×10^{21} tons, Mean density = 5.520 g/cm^3

Ratio of mass of sun to earth = 333,432 : 1

Ratio of mass of earth to moon = 81.45 : 1

Mean distance to sun = 1.497×10^{13} cm = 9.300×10^{17} miles

Distance of sun at perihelion = 1.47×10^{13} cm = 9.136×10^7 miles

Distance of sun at aphelion = 1.52×10^{13} cm = 9.447×10^7 miles

Mean distance to moon = 3.847×10^{10} cm = 2.391×10^5 miles

Number of satellites = 1 (moon)

Greatest height (Mt. Everest) = 29028 ft.

Greatest depth (Challenger Deep) Mariana trench = 35,800 ft.

Lowest on land (Dead sea) = 1286 ft.

Land area = 148.8×10^6 km^2 = 5.747×10^7 miles^2, Ocean area = 361.3×10^6 km^2 = 13.95×10^7 miles^2

Acceleration of gravity (g) in cm per sec per sec. at latitude λ and height (h) (in metres) above sea level
$$g = 980.616 - 2.5928 \cos 2\lambda + 0.0069 (\cos 2\lambda)^2 - 0.0003\, h.$$

Value of g for $\lambda = 45°$ at sea level = 980.621 cm per sec. per sec. = 32.173 ft. per sec. per sec.

Solar energy incident on unit area at right angles to sun's rays at the earth's mean distance per unit time = 2.00 Calories/cm^2/minute.

Age of the earth = Between 4×10^9 and 5×10^9 years

Nearest star (Proxima Centauri) = 4.31 light years

Revolution = 365.256 days, Rotation = 23 hr. 56 min. 4.09 sec.

Rotational velocity of earth at equator = 460 m/s.

Length of seconds pendulum at sea level, latitude 45° = 99.3577 cm = 39.1171 in.

Population in millions (year in brackets) : 1550 (1900), 1907 (1925), 2497 (1950); projections : 3828 (1975), 6267 (2000).

Astronomical Data on Time

1 sidereal day = 86164.0906 mean solar seconds

1 tropical (civil) year = 365.2422 mean solar days, 1 sidereal year = 365.2564 mean solar days

1 anomalistic year = 365.2596 mean solar days

1 synodical month = 29.53059 mean solar days, 1 tropical month = 27.32158 mean solar days, 1 sidereal month = 27.32166 mean solar days.

TABLE 20.2. CONVERSION BETWEEN CENTIGRADE AND FAHRENHEIT

(for a selected range of temperatures)

Centigrade to Fahrenheit						Fahrenheit to Centigrade							
°C	°F	°C	°F	°C	°F	°F	°C	°F	°C	°F	°C	°F	°C
−10	14.0	15	59.0	40	104.0	0	−17.8	65	18.3	90	32.2	115	46.1
− 9	15.8	16	60.8	41	105.8	5	−15.0	66	18.9	91	32.8	116	46.7
− 8	17.6	17	62.6	42	107.6	10	−12.2	67	19.4	92	33.3	117	47.2
− 7	19.4	18	64.4	43	109.4	15	− 9.4	68	20.0	93	33.9	118	47.8
− 6	21.2	19	66.2	44	111.2	20	− 6.7	69	20.6	94	34.4	119	48.3
− 5	23.0	20	68.0	45	113.0	25	− 3.9	70	21.1	95	35.0	120	48.9
− 4	24.8	21	69.8	46	114.8	30	− 1.1	71	21.7	96	35.6	121	49.4
− 3	26.6	22	71.6	47	116.6	35	1.7	72	22.2	97	36.1	122	50.0
− 2	28.4	23	73.4	48	118.4	40	4.4	73	22.8	98	36.7	123	50.6
− 1	30.2	24	75.2	49	120.2	45	7.2	74	23.3	99	37.2	124	51.1
0	32.0	25	77.0	50	122.0	50	10.0	75	23.9	100	37.8	125	51.7
1	33.8	26	78.8	51	123.8	51	10.6	76	24.4	101	38.3	126	52.2
2	35.6	27	80.6	52	125.6	52	11.1	77	25.0	102	38.9	127	52.8
3	37.4	28	82.4	53	127.4	53	11.7	78	25.6	103	39.4	128	53.3
4	39.2	29	84.2	54	129.2	54	12.2	79	26.1	104	40.0	129	53.9
5	41.0	30	86.0	55	131.0	55	12.8	80	26.7	105	40.6	130	54.4
6	42.8	31	87.8	60	140.0	56	13.3	81	27.2	106	41.1	131	55.0
7	44.6	32	89.6	70	158.0	57	13.9	82	27.8	107	41.7	132	55.6
8	46.4	33	91.4	80	176.0	58	14.4	83	28.3	108	42.2	133	56.1
9	48.2	34	93.2	90	194.0	59	15.0	84	28.9	109	42.8	134	56.7
10	50.0	35	95.0	100	212.0	60	15.6	85	29.4	110	43.3	135	57.2
11	51.8	36	96.8	200	392.0	61	16.1	86	30.0	111	43.9	136	57.8
12	53.6	37	98.6	400	752.0	62	16.7	87	30.6	112	44.4	137	58.3
13	55.4	38	100.4	500	932.0	63	17.2	88	31.1	113	45.0	138	58.9
14	57.2	39	102.2	1000	1832.0	64	17.8	89	31.7	114	45.6	139	59.4

The fundamental unit of temparature is degree Kelvin (°K). For purposes of practical measurement the centigrade scale (°C) is internationally adopted. In addition the degree Fahrenheit (°F) and the degree Rankine (°R) are used. The conversions are as shown in the table below.

TEMPERATURE CONVERSION FORMULAE

Systems in degrees	Kelvin (°K)	Centigrade or Celsius (°C)	Fahrenheit (°F)	Rankine (°R)
Kelvin	T_k	$t_c+273.15$	$5(t_f+459.67)/9$	$5T_r/9$
Centigrade	$T_k-273.15$	t_c	$5(t_f-32)/9$	$5(T_r-491.67)/9$
Fahrenheit	$(9T_k/5)-459.67$	$(9t_c/5)+32$	t_f	$T_r-459.67$
Rankine	$9T_k/5$	$(9t_c/5)+491.67$	$t_f+459.67$	T_r

TABLE 20.3. PERIODIC TABLE OF THE ELEMENTS

KEY TO CHART

Atomic Number → 50 +2 +4 ← Oxidation States
Symbol → Sn
Atomic Weight → 118.69
−18–18–4 ← Electron Configuration

1a	2a	3b	4b	5b	6b	7b	8			1b	2b	3a	4a	5a	6a	7a	0	Orbit
1 H +1 −1 1.00797 1																	2 He 0 4.0026 2	K
3 Li +1 6.939 2–1	4 Be +2 9.0122 2–2	Transition Elements										5 B +3 10.811 2–3	6 C +2 +4 −4 12.01115 2–4	7 N +1 +2 +3 +4 +5 −1 −2 −3 14.0067 2–5	8 O −2 15.9994 2–6	9 F −1 18.9984 2–7	10 Ne 0 20.183 2–8	K–L
11 Na +1 22.9898 2–8–1	12 Mg +2 24.312 2–8–2						Group 8					13 Al +3 26.9815 2–8–3	14 Si +2 +4 −4 28.086 2–8–4	15 P +3 +5 −3 30.9738 2–8–5	16 S +4 +6 −2 32.064 2–8–6	17 Cl +1 +5 +7 −1 35.453 2–8–7	18 Ar 0 39.948 2–8–8	K–L–M
19 K +1 39.102 –8–8–1	20 Ca +2 40.08 –8–8–2	21 Sc +3 44.956 –8–9–2	22 Ti +2 +3 +4 47.90 –8–10–2	23 V +2 +3 +4 +5 50.942 –8–11–2	24 Cr +2 +3 +6 51.996 –8–13–1	25 Mn +2 +3 +4 +7 54.9380 –8–13–2	26 Fe +2 +3 55.847 –8–14–2	27 Co +2 +3 58.9332 –8–15–2	28 Ni +2 +3 58.71 –8–16–2	29 Cu +1 +2 63.54 –8–18–1	30 Zn +2 65.37 –8–18–2	31 Ga +3 69.72 –8–18–3	32 Ge +2 +4 72.59 –8–18–4	33 As +3 +5 −3 74.9216 –8–18–5	34 Se +4 +6 −2 78.96 –8–18–6	35 Br +1 +5 −1 79.909 –8–18–7	36 Kr 0 83.80 –8–18–8	–L–M–N
37 Rb +1 85.47 –18–8–1	38 Sr +2 87.62 –18–8–2	39 Y +3 88.905 –18–9–2	40 Zr +4 91.22 –18–10–2	41 Nb +3 +5 92.906 –18–12–1	42 Mo +6 95.94 –18–13–1	43 Tc +4 +6 +7 (99) –18–13–2	44 Ru +3 101.07 –18–15–1	45 Rh +3 102.905 –18–16–1	46 Pd +2 +4 106.4 –18–18–0	47 Ag +1 107.870 –18–18–1	48 Cd +2 112.40 –18–18–2	49 In +3 114.82 –18–18–3	50 Sn +2 +4 118.69 –18–18–4	51 Sb +3 +5 −3 121.75 –18–18–5	52 Te +4 +6 −2 127.60 –18–18–6	53 I +1 +5 +7 −1 126.9044 –18–18–7	54 Xe 0 131.30 –18–18–8	–M–N–O
55 Cs +1 132.905 –18–8–1	56 Ba +2 137.34 –18–8–2	57* La +3 138.91 –18–9–2	72 Hf +4 178.49 –32–10–2	73 Ta +5 180.948 –32–11–2	74 W +6 183.85 –32–12–2	75 Re +4 +6 +7 186.2 –32–13–2	76 Os +3 +4 190.2 –32–14–2	77 Ir +3 +4 192.2 –32–15–2	78 Pt +2 +4 195.09 –32–16–2	79 Au +1 +3 196.967 –32–18–1	80 Hg +1 +2 200.59 –32–18–2	81 Tl +1 +3 204.37 –32–18–3	82 Pb +2 +4 207.19 –32–18–4	83 Bi +3 +5 208.980 –32–18–5	84 Po +2 +4 (242) –32–18–6	85 At (210) –32–18–7	86 Rn 0 (222) –32–18–8	–N–O–P
87 Fr +1 (223) –18–8–1	88 Ra +2 (227) –18–8–2	89** Ac +3 (227) –18–9–2																–O–P–Q
*Lanthanides		58 Ce +3 +4 140.12 –19–9–2	59 Pr +3 140.907 –20–9–2	60 Nd +3 144.24 –22–8–2	61 Pm +3 (145) –23–8–2	62 Sm +2 +3 150.35 –24–8–2	63 Eu +2 +3 151.96 –25–8–2	64 Gd +3 157.25 –25–9–2	65 Tb +3 158.924 –26–9–2	66 Dy +3 162.50 –28–8–2	67 Ho +3 164.930 –29–8–2	68 Er +3 167.26 –30–8–2	69 Tm +3 168.934 –31–8–2	70 Yb +2 +3 173.04 –32–8–2	71 Lu +3 174.97 –32–9–2			–N–O–P
**Actinides		90 Th +4 232.038 –19–9–2	91 Pa +4 +5 (231) –20–9–2	92 U +3 +4 +5 +6 238.03 –21–9–2	93 Np +3 +4 +5 +6 (237) –22–9–2	94 Pu +3 +4 +5 +6 (242) –23–9–2	95 Am +3 +4 +5 +6 (243) –24–9–2	96 Cm +3 (245) –25–9–2	97 Bk +3 +4 (249) –26–9–2	98 Cf +3 (249) –28–8–2	99 Es (254) –29–8–2	100 Fm (252) –30–8–2	101 Md (256) –31–8–2	102 No (254) –32–8–2	103 Lw			–O–P–Q

Numbers in parentheses are mass numbers of most stable isotope of that element.

TABLE 20.4. DENSITY OF VARIOUS SOLIDS[1,2] AND LIQUIDS

solid	density (gms per cu. cm.)	solid	density (gms per cu. cm.)	solid	density (gms per cu. cm.)	liquid	density (gms per cu. cm.)	temp. °C
Agate	2.5—2.7	Diamond	3.01—3.52	Pitch	1.07	Acetone	0.792	20
Aluminium	2.70	Dolomite	2.84	Platinum	21.37	Alcohol, ethyl	0.791	20
Amber	1.06—1.11	Ebonite	1.15	Porcelain	2.3—2.5	methyl	0.810	0
Antimony (compressed)	6.69	Emery	4.0	Quartz	2.65	Benzene	0.899	0
Asbestos	2.0—2.8	Feldspar	2.55—2.75	Resin	1.07			
						Carbolic acid	0.950—0.965	15
Asbestos slate	1.8	Flint	2.63			Carbon, disulfide	1.293	0
Asphalt	1.1—1.5	Gas carbon	1.88			tetrachloride	1.595	20
Basalt	2.4—3.1	Gelatin	1.27			Chloroform	1.489	20
Beryl	2.69—2.7	German Silver	8.5—8.9	Rubber, hard	1.19			
Bismuth	9.80	Glass, common	2.4—2.8	Rubber, soft commercial	1.1	Ether	0.736	0
						Gasoline	0.66—0.69	..
Bone	1.7—2.0	Glass, flint	2.9—5.9	pure gum	0.91—0.93	Glycerin	1.260	0
Brass	8.2—8.8	Glue	1.27	Silica, fused transparent	2.21	Kerosene	0.82	..
Brick	1.4—2.2	Gold	19.3	translucent	2.07	Mercury	13.6	..
Butter	0.86—0.87	Gypsum	2.31—2.33	Silver	10.5	Milk	1.028—1.035	..
Camphor	0.99	Ice	0.917	Sodium	0.97	Naptha, petroleum ether	0.665	15
Carbon (Graphite)	2.25	Invar	8.0	Starch	1.53	wood	0.848—0.810	0
Cardboard	0.69	Ivory	1.83—1.92	Sugar	1.59			
Celluloid	1.4	Iron (cast)	7.0—7.7			Oils:		
Cement, set	2.7—3.0	Iron (wrought)	7.8—7.9					
Chalk	1.9—2.8	Leather, dry	0.86	Sulphur	2.07	castor	0.969	15
Charcoal, oak	0.57	Lime, slaked	1.3—1.4	Talc	2.7—2.8	cocoanut	0.925	15
pine	0.28—0.44	Limestone	2.68—2.76	Tar	1.02	cotton seed	0.926	16
Clay	1.8—2.6	Magnetite	4.9—5.2	Tin	7.3	creosote	1.040—1.100	15
Coal, anthracite	1.4—1.8	Malachite	3.7—4.1					
bituminous	1.2—1.5	Marble	2.6—2.84			linseed, boiled	0.942	15
Cobalt	8.9	Mica	2.6—3.2			olive	0.918	15
Coke	1.0—1.7	Naphthalene	1.15	Tourmaline	3.0—3.2			
Copper (compressed)	8.94	Nickel	8.9	Wax, sealing	1.8	Sea water	1.025	15
Cork	0.22—0.26	Paper	0.7—1.15	Wood (oak)	0.60—0.90	Turpentine (spirits)	0.87	..
Corundum	3.9—4.0	Paraffin	0.87—0.91	Zinc	7.14	Water	1.00	4

[1] At ordinary atmospheric temperature.

[2] In the case of substances with voids such as paper or leather the bulk density is indicated rather than the density of the solid portion.

TABLE 20.5. GEOLOGICAL TIME-SCALE

age in millions of years†	geological systems (maximum thickness in feet)		first appearance of	examples of rock formations
	Quaternary*			
1.5-3.5	PLIOCENE 15,000 ft.	CAENOZOIC	Man, bread, wheat	
7	MIOCENE 21,000 ft.	CAENOZOIC	Most mammalian orders	Siwaliks (in the Himalayas)
26	OLIGOCENE 26,000 ft.	CAENOZOIC	Grass	
37-38	EOCENE. 30,000ft	CAENOZOIC		
53-54	PALEOCENE, 12,000ft	CAENOZOIC		
64-65	CRETACEOUS 51,000 ft.	MESOZOIC	Modern flowering plants Urodeles, Snakes, Marsupials, Insectivores Modern bony fish	Deccan trap
136	JURASSIC 44,000 ft.	MESOZOIC	Flowering plants, Frogs, Plesiosaurs, Pterosaurs, Birds	Rajmahal trap
190-195	TRIASSIC 30,000 ft.	MESOZOIC	Cycads, Ammonites, Modern reptiles (Turtles, Crocodiles, Ichthyosaurs, Dinosaurs)	
225	PERMIAN 19,000 ft.	PALAEOZOIC	Modern insects (Bugs etc.)	Main Indian coal seams (Gondwana)
280	CARBONIFEROUS 46,000 ft.	PALAEOZOIC	Conifers, Gingkos, Reptiles, Winged insects	European coal seams
345	DEVONIAN 38,000 ft.	PALAEOZOIC	More advanced jawed fish (e.g. Sharks), Amphibians, Wingless insects, Spiders	
395	SILURIAN 34,000 ft.	PALAEOZOIC	Land plants, Primitive jawed fish	
430-440	ORDOVICIAN 40,000 ft.	PALAEOZOIC	Corals, Vertebrate fragments of jawless fish	
500	CAMBRIAN 40,000 ft.	PALAEOZOIC	Most invertebrate phyla	Vindhyans
570	Unknown thickness	PRO-TEROZOIC	Algae, Medusae, Annelids, Pennatulids	
	= PRO-CAMBRIAN =			
Origin of Earth's Crust — 5674	Unknown thickness	AZOIC		

* Quarternary (Pleistocene and Holocene), 6,000 feet+

† Radiometric age determinations (after "The Phanerozoic time scale" Geol. Soc., London, 1964)

Adapted from The British Museum (Natural History) series : The Sucession of Life through Geological Time, 1962 with some subsequent revisions.

TABLE 20.6. PROTEIN AND FAT PERCENTAGES AND CALORIES PER 100 GMS. OF FOODSTUFFS

Name of foodstuff	prot. %	fat %	cal.
Milk and milk products			
Milk (Ass)	1.7	1.0	47
Milk (Cow)	3.3	3.6	65
Milk (Buffalo)	4.3	8.8	117
Milk (Goat)	3.7	5.6	84
Milk (Human)	1.0	3.9	67
Curd (Dahi)	2.9	2.9	51
Butter	1.5	85.0	790
Butter milk	0.8	1.1	15
Skimmed milk	2.5	0.1	29
Skimmed milk powder	38.0	0.1	357
Cheese	24.1	25.1	348
Cream	2.5	24.0	245
Casein (channa)	21.5	17.5	252
Sandesh	19.5	20.2	330
Flesh food			
Beef (Muscle)	22.6	2.6	114
Crab (Muscle)	8.9	1.1	60
Eggs (Duck)	13.5	13.7	180
Eggs (Fowl)	13.3	13.3	174
Fish (Rohit)	18.35	7.55	140
Fish (Vetki)	16.25	4.10	105
Fish (Hilsha)	14.85	9.20	150
Fish (Mango)	16.75	4.10	109
Fish (Magoor)	19.50	0.50	85
Fish (Koi)	17.75	0.45	78
Fish (Tangra)	17.30	0.30	72
Fish (Parshe)	15.75	6.2	120
Chicken	21.0	3.0	114
Liver (Sheep)	19.3	7.5	150
Mutton (Muscle)	18.5	13.3	195
Mutton (Lean)	17.0	3.0	100
Mutton (Fat)	11.0	28.0	300
Prawn (Muscle)	20.8	0.3	85

Name of foodstuff	prot. %	fat %	cal.
Cereals			
Bajra or cambu	11.6	5.0	360
Barley	11.5	1.2	335
Cholam	10.4	1.9	341
Maize, tender	4.3	0.5	82
Maize, dry	11.1	3.6	342
Maize, flour	0.6	0.5	355
Oatmeal	13.6	7.6	374
Ragi	7.1	1.3	345
Rice, raw, home-pounded	8.5	0.6	351
Rice, par-boiled, home-pounded	8.5	0.6	349
Rice, raw, milled	6.9	0.4	348
Rice, par-boiled, milled	6.4	0.4	346
Rice, flakes	6.6	1.2	350
Rice, beaten (Chira)	7.8	0.01	344
Rice puffed (Muri)	7.5	0.1	328
Samai	7.7	4.7	328
Fried paddy (Khai)	7.2	0.2	342
Sati flour (Palo)	3.4	3.5	360
Wheat, whole	11.8	1.5	340
Wheat, flour, whole (atta)	12.1	1.7	353
Wheat, flour, refined	11.0	0.9	349
Bread	8.8	1.5	248
Boiled rice (Bhat)	4.8	0.8	215
Chapatí (Atta Ruti)	10.0	1.6	330
Loochi	7.0	22.5	440
Legumes (pulses)			
Bengal gram (with husk)	17.1	5.3	361
Bengal gram, roasted (without husk)	22.5	5.2	372
Black gram (without husk)	24.0	1.4	350
Cow gram	24.6	0.7	327
Field Bean, dry	24.9	0.8	347

TABLE 20.6. (*continued*). PROTEIN AND FAT PERCENTAGES AND CALORIES PER 100 GMS. OF FOODSTUFFS

Name of foodstuff	prot. %	fat %	cal.
Legumes (pulses) (*continued*)			
Green Gram (with husk)	24.0	1.3	334
Horse Gram	22.0	0.5	322
"Khesari"	28.2	0.6	351
Lentil (Masur dal)	25.1	0.7	346
Peas, dried	19.7	1.1	315
Peas, roasted	22.9	1.4	358
Red Gram (Dal arhar) (without husk)	22.3	1.7	333
Soya bean	43.2	19.5	432
Leafy vegetables			
Amaranth, tender	4.9	0.5	47
Amaranth, spined	3.0	0.3	47
Bamboo, tender shoots	3.9	0.5	47
"Bathua" leaves	4.7	0.4	37
Bengal gram leaves	7.0	1.4	87
Brussels sprouts	4.7	0.5	60
Cabbage	1.8	0.1	33
Carrot leaves	5.1	0.5	58
Celery	6.0	0.6	64
Coriander	3.3	0.6	45
Curry leaves	6.1	1.0	97
Drumstick	6.7	1.7	96
Fenugreek	4.9	0.9	67
Garden cress	5.8	1.0	67
Gram leaves	8.2	0.5	146
Ipomoea	2.9	0.4	32
Khesari leaves	6.1	1.0	64
Lettuce	2.1	0.3	23
Mint	4.8	0.6	57
Neem, mature	7.1	1.0	129
Neem, tender	1.6	3.0	158
Parsley	5.9	1.0	111
Rape leaves	5.1	0.4	52
Safflower leaves	3.3	0.7	40
Spinach	1.9	0.9	32
Soya leaves	6.0	0.5	72
Water cress	2.9	0.2	35

Name of foodstuff	prot. %	fat %	cal.
Roots and tubers			
Beet root	1.7	0.1	62
Carrot	0.9	0.1	47
Onion, big	1.2	0.1	51
Onion, small	1.8	0.1	61
Parsnip	1.3	0.3	101
Potato	1.6	0.1	99
Radish (pink)	0.6	0.3	35
Radish (white)	0.7	0.1	21
Sweet potato	1.2	0.3	132
Tapioca	0.7	0.2	159
Yam (elephant)	1.2	0.1	79
Yam (ordinary)	1.4	0.1	115
Other vegetables			
Amaranth stem	0.9	0.1	19
Artichoke	3.6	0.1	79
Ash gourd	0.4	0.1	15
Bitter gourd	1.6	0.2	25
Bitter gourd (small variety)	2.9	1.0	60
Brinjal	1.3	0.3	34
Broad beans	4.5	0.1	59
Calabash cucumber	0.2	0.1	13
Cauliflower	3.5	0.4	39
Celery stalks	0.8	0.1	18
Cluster beans	3.7	0.2	56
Colocasia stems	0.3	0.3	21
Cucumber	0.4	0.1	14
Double beans	8.3	0.3	85
Drumstick	2.5	0.1	26
French beans	1.7	0.1	26
Ipomoea stems	0.9	0.2	19
Jack, tender	2.6	0.3	51
Jack fruit seeds	6.6	0.4	134
"Kovai" fruit, tender	1.2	0.1	20

TABLE 20.6. (*continued*). PROTEIN AND FAT PERCENTAGES AND CALORIES PER 100 GMS. OF FOODSTUFFS

Name of foodstuff	prot. %	fat %	cal.
Other vegetables (*continued*)			
Knol-knol	1.1	0.2	30
Ladies fingers	2.2	0.2	41
Leeks	1.8	0.1	77
Mango, green	0.7	0.1	39
'Nellikai'' Amla	0.5	0.1	59
Onion stalks	0.9	0.2	41
"Parwar"	2.0	0.3	18
Peas, English	7.2	0.1	109
Plantain flower	1.5	0.2	28
Plaintain, green	1.4	0.2	66
Plantain stem	0.5	0.1	42
Pumpkin	1.4	0.1	28
Rhubarb stalks	1.1	0.5	24
Ridge gourd	0.5	0.1	18
"Singhara" or water chestnut	4.7	0.3	117
Snake-gourd	0.5	0.3	22
Spinach stalks	0.9	0.1	20
Sword beans	2.7	0.2	38
Tomato, green	1.9	0.1	27
Turnip	0.5	0.2	34
Vegetable marrow	0.5	0.1	20
Fruits			
Apple	0.9	0.1	56
Banana	1.3	0.2	153
Billimbi	0.5	0.2	23
Cashew fruit	0.2	0.1	48
Dates (Persian)	3.0	0.2	283
Figs (fresh)	1.3	0.2	75
Grapes (Blue Variety)	0.8	0.1	45
Grape fruit (Trimph)	0.7	0.1	32
Grape fruit (Marsh's seedless)	1.0	0.1	45
Guava, country	1.5	0.2	66
Guava, hill	0.1	0.2	38
Jack fruit	1.9	0.1	84
Jambu fruit (Rose apple)	0.7	0.1	3
Korukkapalli	2.6	0.3	77
Lemon	1.0	0.9	57
Fruits (*continued*)			
Lime	1.5	1.0	59
Mango, green	0.7	0.1	39
Mango, ripe	0.6	0.1	50
Mango, "Ankola"	1.0	0.1	55
Mangosteen	0.5	0.1	60
Melon, water	0.1	0.2	17
Orange	0.9	0.3	49
Palmyra fruit, tender	0.6	0.1	23
Papaya, ripe	0.5	0.1	40
Peaches	1.5	0.2	38
Pears, country	0.2	0.1	47
Pears, English	0.9	0.2	57
Pineapple	0.6	0.1	50
Plaintain (ordinary)	1.1	0.1	104
Plaintain (red variety)	1.6	0.1	101
Plums (red variety)	0.7	0.2	40
Pomegranate	1.6	10.1	65
Pomeloe	0.6	10.1	44
Quince	0.3	0.1	49
Radish fruit	2.3	0.3	34
Raisins (preserved)	2.0	0.2	319
"Seetha Pazham" or Custard apple	1.6	0.3	105
Strawberry	0.7	0.2	44
Tomato, ripe	1.0	0.1	21
"Vikki Pazham" or Wild olive	1.4	0.1	141
Wood apple	7.3	0.6	97
Tamarind, pulp	3.1	0.1	283
Zizyphus (Indian plum)	0.8	0.1	55
Nuts and oil seeds			
Almond	20.8	58.9	655
Cashew nut	21.2	46.9	596
Cocoanut	4.5	41.6	444
Gingili seeds	18.3	43.3	564
Ground nut	26.7	40.1	549

TABLE 20.6. (*continued*). PROTEIN AND FAT PERCENTAGES AND CALORIES PER 100 GMS. OF FOODSTUFFS

Name of foodstuff	prot. %	fat %	cal.
Nuts and oil seeds (*continued*)			
Ground nut roasted	31.5	39.8	561
Linseed seeds	20.3	37.1	530
Mustard seeds	22.0	39.7	541
Pistachio nut	19.8	53.5	626
Walnut	15.6	64.5	687
Miscellaneous foodstuffs			
Arecanut	4.9	4.4	248
Arrow-root flour (West Indian)	0.2	0.1	334
Betel leaves (piper betel)	3.1	0.8	44
Cocoanut, tender	0.9	1.4	40
Cocoanut, water	0.1	0.1	17
Cooking oil	...	98.0	895
Cod liver oil	...	100.0	900
Halibut liver oil	...	100.0	900
Honey	0.5	...	325
Jaggery (Gur)	0.4	0.1	383
Jam	0.3	...	315
"Makhana"	9.7	0.1	348
Red palm oil	...	100.0	900
Sago	0.2	0.2	351
Sugar	...	...	390

Name of foodstuff	prot. %	fat %	cal.
Miscellaneous foodstuffs (*continued*)			
Sugar, cane juice	0.1	0.2	39
Sugar cane preserves	0.6	0.1	317
Toddy, sweet	0.1	0.2	59
Toddy, sweet (cocoanut)	0.1	0.1	15
Toddy, fermented (cocoanut)	0.2	0.1	7
Yeast, dried	39.5	0.6	320
Condiments, spices, etc.			
Asafoetida	4.0	1.1	297
Cardamom	10.2	2.2	229
Cloves, dry	5.2	8.9	293
Cloves, green	2.3	5.9	159
Coriander	14.1	16.1	288
Cumin	18.7	15.0	356
Fenugreek seeds	26.2	5.8	333
Garlic	6.3	0.1	142
Ginger	2.3	0.9	67
Kandanthippili (Long pepper)	6.4	2.3	310
Lime peel	1.8	0.5	129
Mace	6.5	24.4	437
Mustard	22.0	39.7	541
Nutmeg	7.5	36.4	472
Omum	15.4	18.1	379
Pepper, green	4.8	2.7	153
Pepper, dry	11.5	6.8	305
Turmeric	6.3	5.1	349

Sources

(i) *Food and Nutrition in India*, published by Dr. D. N. Chatterji, Calcutta, 1947

(ii) *Our Food* by M. Swaminathan and R. K. Bhagawan. Ganesh & Co., Madras, 1959

PROOF CORRECTION GUIDE

Specimen of proof sheet with corrections

4. THE POISSON distribution

4.1 INDIVIDUAL terms

¶/Cap/⌴/ 1. table 4. 1 gives values of $p(x, \lambda) = e^{-\lambda}\lambda^x/x!$, $x = 0, 1, 2, \dots$
for λ=0.1 (0.1) 1.0, 1.5, 2.0 (1.0) 10.0.

no ¶/ The values are correct to eight places of decimal for λ upto w.f./

out s.c./ 2. For purposes of (λ-wise) interpolation between the tabula-

eq # ted values the following formula based on Taylor expansion will

⊏/ l.c/ be found useful. Let the Value of $p(x, \lambda)$ be required for a λ. (Greek lamda)/

n/ given λ and λ_0 stand for the tabular argument closest to λ. rom/

Write d=$\lambda-\lambda_0$.

Then,

$$p(x, \lambda) = p(x, \lambda_0) - d\Delta_x p(x-1, \lambda_0) + \frac{d}{2!}\, \Delta_x^2 p(x-2, \lambda_0) + \dots$$ 2/

centre $\langle +(-1)^k \frac{d^k}{k!} \Delta^k p(x-k, \lambda_0) + \dots \rangle$,/ 9/

where Δ_x, Δ_x^2, ... are the 1st, 2nd, .. order differencestaken with #/

stet ~~respect~~ to x, and $R = \frac{d^{k+1}}{(k+1)!} \Delta_x^{k+1}\, p(x-k-1, \lambda^*)$, where λ^* is

some value lying between λ_0 and λ . It will thus be possibble ↓ δ/

by ~~inspection~~ of the tabulated values to judge the maximum ⊙

=/ ⌒/ possible mag nitude for the error R

ital/ tr/ Example. λ=5.32, x=3. ital/

PROOFREADER'S MARKS

MARK	MEANING	MARK	MEANING	MARK	MEANING
Cap	Capital letter	δ	Delete	e/	Substitute e for the letter struck off
l.c.	Lower-case letter	⌒	Close up	↓	Push down quad
^,	Insert comma	tr	Transpose	eq #	Equalize spacing
×	Fix broken letter	⊏/	Move left	stet	Let type stand
#	Insert space	⊐/	Move right	w.f.	Change to right font
9	Invert letter	⌐	Raise	¶	Begin new paragraph
ᵛ,,	Insert quotes	⌴	Lower	no ¶	No paragraph, run in
δ̂	Delete and close up	< >	Centre	=/	Insert hyphen

Specimen of proof sheet after correction

4. THE POISSON DISTRIBUTION

4.1. Individual Terms

1. Table 4.1 gives values of $p(x, \lambda) = e^{-\lambda}\, \lambda^x/x!$, $x = 0, 1, 2, \ldots$ for $\lambda = 0.1\ (0.1)\ 1.0, 1.5, 2.0\ (1.0)\ 10.0$. The values are correct to eight places of decimal for λ upto 5.0 and to seven places of decimal for $\lambda = 6.0$ to 10.0.

2. For purposes of (λ-wise) interpolation between the tabulated values the following formula based on Taylor expansion will be found useful. Let the value of $p(x, \lambda)$ be required for a given λ and λ_0 stand for the tabular argument closest to λ. Write $d = \lambda - \lambda_0$. Then,

$$p(x, \lambda) = p(x, \lambda_0) - d\Delta_x\, p(x-1, \lambda_0) + \frac{d^2}{2!}\Delta_x^2\, p(x-2, \lambda_0) + \ldots$$

$$+(-1)^k \frac{d^k}{k!}\Delta_x^k\, p(x-k, \lambda_0) + R.$$

where $\Delta_x, \Delta_x^2, \ldots$ are the 1st, 2nd, ... order differences taken with respect to x, and $R = \dfrac{d^{k+1}}{(k+1)!}\Delta_x^{k+1}\, p(x-k-1,\ \lambda^*)$, where λ^* is some value lying between λ_0 and λ. It will thus be possible by inspection of the tabulated values to judge the maximum possible magnitude for the error R.

Example $\lambda = 5.25,\ x = 3$.

PROOFREADER'S MARKS (*contd.*)

MARK	MEANING	MARK (in text)	MEANING
⊙	Insert full stop		
s.c.	Set in small caps		Set in caps
ital.	Set in italics		Set in small caps
rom.	Set in roman		Set in italics
	Straighten line		Set in bold type
∨	Superior figure		Set in bold caps
∧	Inferior figure		Set in bold small caps
□	Em quad space		Set in bold italics
out s.c.	Out see copy (be sure manuscript is returned if this is used)		

ROMAN AND HINDI NUMERALS

a. Roman numerals

The system invented by the early Romans about 2000 years ago was widely used by the people of Europe until about the 16th century. Roman numerals are still used on clocks and monuments, to show chapters of a book, and for volume numbers of some journals.

The Roman system is built on the base of ten and uses the symbols :

I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000. The first twenty numbers are as follows :

I = 1	VI = 6	XI = 11	XVI = 16
II = 2	VII = 7	XII = 12	XVII = 17
III = 3	VIII = 8	XIII = 13	XVIII = 18
IV = 4	IX = 9	XIV = 14	XIX = 19
V = 5	X = 10	XV = 15	XX = 20

There are two rules of writing numbers. (1) If a letter or a set of letters is placed before a letter of higher value, it is to be subtracted from the latter. Thus IV = 4, XC = 90. (2) If a letter of smaller value is placed after one of larger value it is to be added. Thus LX = 60, LV = 55. The Romans first read the thousands, then the tens, then the ones. To read numbers, sometimes one counts, as in counting III, sometimes subtracts, as in finding the value of IV, sometimes adds as in finding the value of XVIII. Thus

MCM XX = 1,920, CCCXLVI = 346

MDC XXXVIII = 1,638, MMMM = 4,000

A line drawn above a group of letters multiplies the number by one thousand. Thus

$$\overline{\text{MDC XXXVIII}} = 1{,}638{,}000.$$

b. Devanagari (Hindi) numerals

० = 0, १ = 1, २ = 2, ३ = 3, ४ = 4,

५ = 5, ६ = 6, ७ = 7, ८ = 8, ९ = 9.

ALPHABETS OF GREEK, GERMAN, HEBREW, RUSSIAN AND HINDI LANGUAGES

GREEK

Α α	alpha	ah
Β β	beta	b
Γ γ	gamma	g
Δ δ	delta	d
Ε ε	epsilon	ĕ
Ζ ζ	zeta	z
Η η	eta	āy
Θ θ	theta	th
Ι ι	iota	ĭ, ee
Κ κ	kappa	k
Λ λ	lambda	l
Μ μ	mu	m
Ν ν	nu	n
Ξ ξ	Xi	x
Ο ο	omicron	o
Π π	pi	p
Ρ ρ	rho	r
Σ σ, ς	sigma	s
Τ τ	tau	t
Υ υ	üpsilon	ü
Φ φ	phi	ph, f = f
Χ χ	chi	K
Ψ ψ	psi	ps
Ω ω	omega	ō

GERMAN

𝔄 𝔞	ah
𝔅 𝔟	b
ℭ 𝔠	k, ts, s
𝔇 𝔡	d
𝔈 𝔢	ĕ, āy
𝔉 𝔣	f
𝔊 𝔤	g
ℌ 𝔥	h
ℑ 𝔦	ĭ, ee
𝔍 𝔧	y (as in *yes*)
𝔎 𝔨	k
𝔏 𝔩	l
𝔐 𝔪	m
𝔑 𝔫	n
𝔒 𝔬	ō
𝔓 𝔭	p
𝔔 𝔮	k
ℜ 𝔯	r
𝔖 𝔰	s
𝔗 𝔱	t
𝔘 𝔲	oo
𝔙 𝔳	f, w
𝔚 𝔴	v
𝔛 𝔵	x
𝔜 𝔶	y
ℨ 𝔷	z, ts

HEBREW

א	aleph	—
ב	bet	b, bh
ג	gimel	g or gh
ד	dalet	d, *th*
ה	he	h
ו	vaw	w, v
ז	zayin	z
ח	het	K
ט	tet	t
י	yod	y (as in *yet*)
כ	kaph	k or K
ל	lamed	l
מ	mem	m
נ	nun	n
ס	samek	s
ע	'ayin	—
פ	pe	p, ph
צ	sade	s or ts
ק	koph	k
ר	resh	r
ש	shin	sh
ת	tav	t, th

RUSSIAN

А а	ah
Б б	b
В в	v
Г г	g
Д д	d
Е е	y
Ж ж	zh
З з	z
И и	ee
К к	k
Л л	l
М м	m
Н н	n
О о	aw, ah
П п	p
Р р	r
С с	s
Т т	t
У у	oo
Ф ф	f
Х х	K
Ц ц	ts
Ч ч	ch, ts
Ш ш	sh
Щ щ	sh + ch
ъ	(apostrophe)
ы	ĭ, wee
ь	(a soft consonant)
Э э	ĕ
Ю ю	yoo
Я я	yah

HINDI

अ	a	ड, ड़	ḍa
आ	ā	ढ, ढ़	ḍha
इ	i	ण	ṇa
ई	ī	त	ta
उ	u	थ	tha
ऊ	ū	द	da
ऋ	r	ध	dha
ए	e	न	na
ऐ	ai	प	pa
ओ	o	फ	pha
औ	au	ब	ba
क	ka	भ	bha
ख	kha	म	ma
ग	ga	य	ya
घ	gha	र	ra
ङ	ṅa	ल	la
च	ca	व	va
छ	cha	श	śa
ज	ja	ष	ṣa
ज़	jạ	स	sa
झ	jha	ह	ha
ञ	ña	ं	ṁ
ट	ṭa	ँ	ṁ
ठ	ṭha	ः	ḥ

PERPETUAL CALENDAR

Code Numbers of Years : 1600—2000

(Code numbers are in roman numerals and for years only tens and units are recorded, the hundreds being indicated at the top. Thus the code number of 1616 is V, of 1920 is IV and so on.)

Years 1600–1699

I	II	III	IV	V	VI	VII
01	02	03	**04**	10	**00**	06
07	**08**	14	09	**16**	05	**12**
18	13	**20**	15	21	11	17
24	19	25	26	27	22	23
29	30	31	**32**	38	**28**	34
35	**36**	42	37	**44**	33	**40**
46	41	**48**	43	49	39	45
52	47	53	54	55	50	51
57	58	59	**60**	66	**56**	62
63	**64**	70	65	**72**	61	**68**
74	69	**76**	71	77	67	73
80	75	81	82	83	78	79
85	86	87	**88**	94	**84**	90
91	**92**	98	93		89	**96**
	97		99		95	

Years 1700–1799

I	II	III	IV	V	VI	VII
03	**04**	10	05	00	01	02
14	09	**16**	11	06	07	**08**
20	15	21	22	**12**	18	13
25	26	27	**28**	17	**24**	19
31	**32**	38	33	23	29	30
42	37	**44**	39	34	35	**36**
48	43	49	50	**40**	46	41
53	54	55	**56**	45	**52**	47
59	**60**	66	61	51	57	58
70	65	**72**	67	62	63	**64**
76	71	77	78	**68**	74	69
81	82	83	**84**	73	**80**	75
87	**88**	94	89	79	85	86
98	93		95	90	91	**92**
	99			**96**		97

Years 1800–1899

I	II	III	IV	V	VI	VII
10	05	00	01	02	03	**04**
16	11	06	07	**08**	14	09
21	22	**12**	18	13	**20**	15
27	**28**	17	**24**	19	25	26
38	33	23	29	30	31	**32**
44	39	34	35	**36**	42	37
49	50	**40**	46	41	**48**	43
55	**56**	45	**52**	47	53	54
66	61	51	57	58	59	**60**
72	67	62	63	**64**	70	65
77	78	**68**	74	69	**76**	71
83	**84**	73	**80**	75	81	82
94	89	79	85	86	87	**88**
	95	90	91	**92**	98	93
		96		97		99

Years 1900–1999

I	II	III	IV	V	VI	VII
00	01	02	03	**04**	10	05
06	07	**08**	14	09	**16**	11
12	18	13	**20**	15	21	22
17	**24**	19	25	26	27	**28**
23	29	30	31	**32**	38	33
34	35	**36**	42	37	**44**	39
40	46	41	**48**	43	49	50
45	**52**	47	53	54	55	**56**
51	57	58	59	**60**	66	61
62	63	**64**	70	65	**72**	67
68	74	69	**76**	71	77	78
73	**80**	75	81	82	83	**84**
79	85	86	87	**88**	94	89
90	91	**92**	98	93		95
96		97		99		

1. Code number of the year 2000 is VI.

2. A leap year is one which is divisible by 4, except that in the case of a century it should be divisible by 400. Thus 1900 is not a leap year but 2000 is.

3. The code numbers are based on the Gregorian calendar which was first adopted in 1582.

4. Leap years are printed in bold face.

PERPETUAL CALENDAR
(GREGORIAN)

NON LEAP YEAR	I	II	III	IV	V	VI	VII	LEAP YEAR
	Days of the Week							
APR, JULY	Su	M	T	W	Th	F	Sa	SEP, DEC
JAN, OCT	M	T	W	Th	F	Sa	Su	JAN, APR, JULY
MAY	T	W	Th	F	Sa	Su	M	OCT
AUG	W	Th	F	Sa	Su	M	T	MAY
FEB, MAR, NOV	Th	F	Sa	Su	M	T	W	FEB, AUG
JUNE	F	Sa	Su	M	T	W	Th	MAR, NOV
SEP, DEC	Sa	Su	M	T	W	Th	F	JUNE
	1	**2**	**3**	**4**	**5**	**6**	**7**	
	8	**9**	**10**	**11**	**12**	**13**	**14**	
	15	**16**	**17**	**18**	**19**	**20**	**21**	
	22	**23**	**24**	**25**	**26**	**27**	**28**	
	29	**30**	**31**					

(Column headers I–VII: CODE NUMBER OF YEAR)

To find the calendar for a given year and month there are three steps.

(1) Find the code number of the given year from the previous page.

(2) If it is a Leap year (in bold) use the months on the right; if not, the months on the left of the above Table. Read the day of the week corresponding to the given month and code number of given year as found in (1).

(3) Observe that there are 7 rows of the days of the week. Choose that row beginning with the day of the week as determined in (2). This row together with the bottom portion of the Table containing the dates from 1 to 31 provides the calendar for the given month and year.

Hold the index fingure of the left hand against the chosen row (of the days of the week) and read the day of the week corresponding to any given date.

Example : What day of the week was June 29, 1893 ?

Code number of 1893 is VII. Using the months for a nonleap year, the day of the week for June and year code VII is Th (Thursday). Then using the row beginning with (Th) we find that 29th was Thursday.

Verify that 10 September 1632 was Friday.

INDEX

NOTES

NOTES

NOTES

NOTES